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Keywords: *approximation, interpolation, piecewise linear approximation, difficult-to-compute function, efficiency index.*

UDC 518: 517.9

MODERNIZATION ADAPTIVE PIECEWISE LINEAR APPROXIMATION OF DIFFICULT-TO-COMPUTE FUNCTIONS

An adaptive approach to the approximation of a continuous one-dimensional function, using a piecewise linear approximation, is considered. A simple mechanism of adaptive control of the stepping process with feedback is used. The possibilities of the approach are considered on the problems of calculating lengths of curves and values of certain integrals. The results of calculating definite integrals with a different character of the integrand obtained by both the proposed method and the usual trapezoidal method are presented. Numerical results showed high efficiency of the proposed adaptive approach.

Introduction

The solution of many theoretical and applied problems requires that some functional dependencies $f(x)$ be substituted into other $g(x)$, which are more convenient for the realization of a specific problem. Such a substitution occurs when it is assumed that a number of requirements for the approximating object $g(x)$, its form and properties are specified. These requirements include: class of functions, accuracy of approximation, computational costs, form of the final result, and much more. To describe the characteristics of such a substitution, the concept of the quality criterion [1] is introduced, which takes into account the accuracy achieved with respect to the number of computations spent for the functions $f(x)$. We mean the aggregate computations of the function $f(x)$, previously spent on analyzing its belonging to a particular class, the actual costs of the approximation $g(x)$ to $f(x)$, as well as the cost of recomputation, in case of failure to immediately achieve the desired result. In such a case, a priori information about the character (class) of the initial function may be missing, and the function itself fall into the category of difficult-to-compute ones. Here we have to use rather conventional, sometimes subjective concepts, such as poorly organized, difficult-to-compute functions, sometimes associated with poor conditionality [2]. Often, these are cases when large computational costs are required to obtain the value of the function itself, associated with solving large systems of differential or non-linear equations.

In Figs. 1, 2 are shown the fragments of functions $f(x)$ specified on the segment $[A, B]$ and consisting of a series of parts, each of which has different singularities, which must be taken into account in the approximation. In this case, it is necessary to clearly distinguish between the methods of specifying functions − coordinate (Fig. 1) and parametric (Fig. 2), which can complicate the numerical implementation.

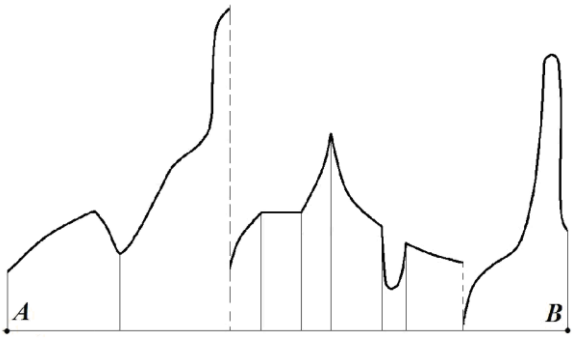
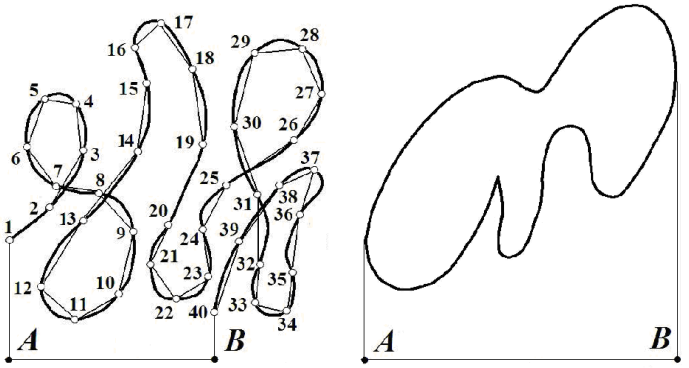


Fig. 1. Coordinate method of defining functions



a b

Fig. 2. Parametric method of defining functions
a – self-intersection; b –− closed form

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According to the Weierstrass theorem [3–5], any continuous function $f(x)$ of class C^2 can be uniformly approximated as closely as desired by polynomials. There are known classical Newton or Lagrange methods [2]

$$g_n(x) = \sum_{i=1}^{n} f(x_i) \frac{\omega_n(x)}{(x - x_i)\omega_n'(x_i)}, \qquad \omega_n(x) = \prod_{i=0}^{n} (x - x_i), \tag{1}$$

which allow us to approximate the original function with a specified accuracy ε. In this case, the achieved accuracy of ε depends on the number of grid points

$$A = x_0, x_1, \ldots, x_n = B. \tag{2}$$

Initially, it is difficult to determine how many grid points (2) will be required for an approximation (1), so that the requirement

$$R_n = \left| f(x) - g_n(x) \right| < \varepsilon, \quad \forall x \in [A, B], \tag{3}$$

will be fulfilled with the specified precision ε.

Therefore, a small number n is taken first, and if it does not suit, it is increased. For example, in the L.F. Richardson scheme [6], the existing grid is divided by additional internal points, which increases their number by about a factor of two (C. Runge principle [7]). Whether the result is satisfactory becomes clear only after a number of systems of equations have been solved. Depending on the properties of the function $f(x)$, the degree of the polynomial (1) can be so great that even modern computers are not capable of resolving the necessary system of equations at a reasonable time. In practice, the number of equations is often limited to not more than one hundred.

Such difficulties can sometimes be overcome by a special choice of interpolation points. This requires a preliminary analysis of the function $f(x)$ being approximated and, therefore, additional computational costs.

The interpolation error (3) can be represented in the form

$$R_n(x) = \left| f(x) - g_n(x) \right| = \frac{f^{(n+1)}}{(n+1)!} \Pi_{n+1}(x), \quad \xi \in [A, B].$$

Here one part depends on the properties of the function $f(x)$ and is not subject to regulation, whereas the other one is related to the distribution of the interpolation points. Using this, you can improve the grid (2) by specifying points according to the formula

$$x_i = 0.5[B + A + \xi_i (B - A)], \tag{4}$$

$$\xi_i = -\cos[\pi(2i + 1)/(2n + 2)], \quad i = 0, 1, 2, \ldots, n,$$

where ξ_i are the zeros of Chebyshev polynomial of the first kind [4, 5, 8]

$$T_{n+1}(x) = \cos\left[(n + 1) \arccos x\right], \quad x \in \left[-1, 1\right].$$

But even with such an optimal case of generating points (4), one can not always be sure that the absolute value of the error will be as small as desired at sufficiently large values of n.

The G. Faber theorem [9] claims that, no matter what the grid is (2), we can select a continuous function $f(x)$ on $[A, B]$ such that the sequence of the interpolation polynomials $\{g_n(x)\}$ does not converge with $f(x)$. Of course, for the function $f(x)$ one can achieve convergence due to a special arrangement of the grid points (2). This is possible by virtue of the J. Marcinkiewicz theorem [10], which claims that for a continuous function $f(x)$ on $[A, B]$ there is always a sequence of points for which the corresponding interpolation process coincides. But almost all such grids are extremely difficult to build in order to be included into a standard algorithm, i.e. each function needs its own grid.

C. Runge in [7] showed that the interpolation process of the approximation $\{g_n(x)\}$, even for a relatively simple and arbitrary differentiable function $f(x) = [1 + (5x)^2]^{-1}$, does not converge on the segment [-1; 1]. Later S.N. Bernstein in [11] paid attention to the fact that a simple continuous function $f(x) = |x|$ on the same interval can not be approximated by the sequence $\{g_n(x)\}$, that is, it turns out that

$$\lim_{n \to \infty} \max_{-1 \le x \le 1} \left| |x| - g_n(x) \right| = \infty.$$

In this, a high degree of the interpolating polynomial can often lead to the accumulation of rounding errors. Therefore, in practice, using high-degree interpolation polynomials is avoided, since the interpolation error increases in proportion to the degree of the polynomial.

The way out of this situation was partly the transition to piecewise polynomial interpolation with a low-degree polynomial. In this case, the segment [*A*, *B*] is divided into sub-segments and on each of them the function $f(x)$ is approximated by low-degree polynomials. One such effective method of approximating functions, which has become widely used in computational practice, is that of spline interpolation [12], based on the use of the simplest forms, in particular, on third and fifth degree polynomials [13], which are most convenient for applied problems. Naturally, increasing the accuracy of spline approximation can go both along the path of increasing the degree of the polynomial $g(x)$ and the choice of a special law for inputting an irregular grid of arguments. At that, it should be taken into account that if the initial function has singularities, such as discontinuities, then the higher is the order of the spline is, the worse the data will be interpolated in the vicinity of the singularity.

One of the ways to improve this can be the use of one or another modification of the spline method [12–15], especially since there are plenty of variants for the present time and new ones are being developed. For example, various basic splines appeared on the basis of the balanced approximations of Popov [13], the atomic functions of V.L. Rvachev [14], the local splines of V.S. Ryabenky [15], the barycentric interpolation [16], and many others. All of them are oriented on the theoretical side of approximation, rather than on the applied one, that is, they take into account the features of the function being approximated, specifying this each time with different requirements for it. In this case, the idea of approximation in itself is of a purely applied nature.

Various modifications aimed at improving the methods of approximation turn out to be more complex both algorithmically and programmatically, and therefore more costly. At the same time, a natural desire to improve approximation methods and increase their potential capabilities, entails, as a rule, a narrowing of the class of approximatable functions.

The aim of the paper is to create effective adaptive methods for the piecewise-linear approximation of functions for the problem of finding the lengths of curves and calculating integrals under the conditions of limited information about the nature of the function itself and the presence of its derivatives.

1. Adaptation

Most often the information about the function $f(x)$ being approximated is obtained using equidistant points. However, in different parts of the segment [*A*, *B*], the behavior of the function can be very different (see Fig. 1, 2). Therefore, the composition must correspond to the nature of the change in the function. Of course, even the optimal distribution of the points (4) will be the best only for the specific case of the function $f(x)$. In this case, as a rule, it is necessary to solve a system of nonlinear equations. Such an approach can hardly be justified as a standard one.

There are two approaches to constructing grids: a priori and a posteriori. Historically, almost all the considered approximation problems and methods for solving them relied on the first approach, which inevitably leads to a narrower field of their application.

In the second approach, a grid with the desired properties is constructed step by step on the basis of a relatively small volume of initial and current information, the properties being necessary to approximate a wide class of functions, including difficult ones. This turns out to be possible due to the inclusion of adaptive control into the adjustment process, in which case it is responsible for the accuracy of the approximation [17–24]. This approach is uniform for different approximatable functions and proceeds in automatic mode. Unlike traditional methods, it makes it possible to successfully solve many problems by very simple means which contribute to improving the efficiency of approximation, without worrying about a preliminary clarification of the characteristic features of the function being approximated.

Adaptive control *U* is usually understood as non-optimal control in a system with incomplete a priori information, but compensated by the accumulated information about the process and used to improve the quality of the system operation [25].

The adaptive scheme for solving an approximation problem can be based on constructing the following grid point x_{k+1} (2) from the previous one x_k

$$x_{k+1} = x_k + h_k \,, \tag{5}$$

where h_k is the adaptively changing step

$$h_{k+1} = h_k U(Q_k) \,. \tag{6}$$

The a posteriori criterion for the situation Q_k included into (6), which in a certain sense characterizes the state of the approximation process formed at the moment k, is in general a multidimensional functional on the set of signs $\{Q_k\}$ of the situation Q. The signs can be either various direct or derivative measurable characteristics of the process (6), for example, performance, computational costs, local accuracy of the approximation obtained, or the degree of its deviation from a given value, etc. [17, 21–25].

Under rather general conditions, the steps of the process (5) can be regarded as positive, which implies the positivity of the function $U(Q)$. We define the function $U(Q)$ on the whole axis $-\infty< Q <\infty$. It follows from (6) that an increase (decrease) in $U(Q)$ results in an increase (decrease) in a step. We specialize the function $U(Q)$, requiring an inverse proportionality between the values of the function $U(Q)$ at points equidistant from the position of the initial situation. This is caused by the fact that the decrease or increase in a step should occur in the same measure for equal in absolute value, but different-in-sign changes in Q [20].

These properties are possessed by a unique function

$$U(Q) = \exp(\alpha(Q)),$$

where α is the adaptation coefficient characterizing the intensity of the step process (6)

$$\alpha = \ln C, \quad C = U(1) > 0.$$

As a result, the law of adaptive control of the step (6) can be represented by the formula

$$h_{k+1} = h_k \exp(\alpha Q_k). \tag{7}$$

It is convenient to write the situation Q_k as the level of deviation of the function $g(x)$ from the function $f(x)$

$$Q_k = \max_{x_k < x < x_k} \| f(x) - g(x) \|. \tag{8}$$

If it is necessary to keep the deviation (8) within certain limits, then it is natural to add into (7) the difference

$$\varepsilon - Q_k \tag{9}$$

and obtain the law of adaptive step control in the form

$$h_{k+1} = h_k \exp[\alpha(\varepsilon - Q_k)], \tag{10}$$

where ε is some degree of admissible deviation.

On the basis of the adaptive mechanism (10) it is possible to solve specific problems taking into account the peculiarities inherent to them. We illustrate the possibilities of this approach of approximation of functions on the problems of calculating the length of arcs of curves and definite integrals.

2. Adaptive approach in the problem of determining the length of a curve line

One of the important numerical characteristics of a line is its length. If a line segment is a curve, then its length is the limit of consecutively inscribed broken lines in the line, when the number of links increases indefinitely, the length of the largest of them tending to zero (see Figure 2, a). We will approximately measure the length of a curve by the sum of the lengths of the inscribed links of the broken line $l_1, l_2, \ldots, l_n$

$$L = \sum_{k=1}^{n} l_k, \tag{11}$$

where the length of each link l_k in rectangular coordinates can be written as

$$l_k = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}. \tag{12}$$

According to the adaptive law (10), the process of approximating the length L of the curve $y=f(x)$ by $[A, B]$ is carried out step by step (Fig. 3) by the formula

$$h_{k+1} = h_k \exp[\alpha(\varepsilon - \left| y_k^* - g_k^* \right|)], \quad g_k^* = (y_k + y_{k+1})/2, \tag{13}$$

where y_k^* is the approximate value of the function $f(x)$ at the midpoint $x_k^* = (x_k + x_{k+1})/2$, the most probable maximum deviation of the function $g(x)$ from $f(x)$.

Naturally, in the formula (13) one can use not the approximate value of $f(x)$ but the exact one. However, then we will need to calculate the additional value of the function $f(x)$ at the midpoint, which we want to avoid.

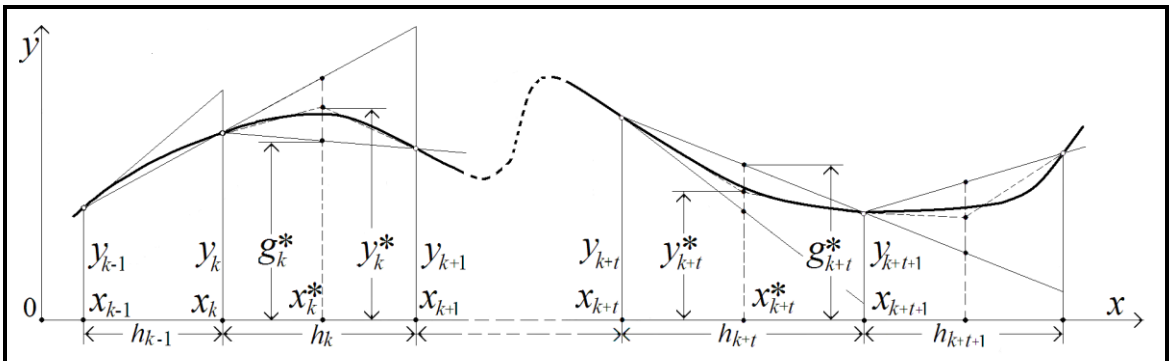

Fig. 3. Fragment of the deviation scheme for g from f for adaptive step control

From Fig. 3 it is clear that the true ordinate of the function $f(x)$, determined at the midpoint x_k^*, can be approximately written as follows:

$$y_k^* = 0.25[y_{k+1} + (3+\mu_k)y_k - \mu_k y_{k+1}], \quad \mu_k = h_k / h_{k-1}. \tag{14}$$

In this case, not one (solid line) is added to the current total length of the polyline, but the lengths of two chords (dashed lines), which approximate the curve segment $y=f(x)$ on the segment $[x_k, x_{k+1}]$ (Fig. 3) i.e. instead of (12)

$$l_k = \sqrt{\left(x_k - x_k^*\right)^2 + \left(y_k - y_k^*\right)^2} + \sqrt{\left(x_{k+1} - x_k^*\right)^2 + \left(y_{k+1} - y_k^*\right)^2}$$

is applied and then the direction set by the chord $y_k\ y_{k+1}$ is substituted into $y_k^*\ y_{k+1}$.

Sometimes complex curves, especially those with self-intersections (see Fig. 2, a), including closed–form ones (see Fig. 2, b), are given by formulas in the parametric representation. If a curve on the interval $[t_k, t_{k+1}]$ is given in the form $x=x(t)$, $y=y(t)$, $t\in[t_k, t_{k+1}]$, the length of a link is determined by the expression

$$l_k = \sqrt{\left[(x_k - x_{k+1})/(t_k - t_{k+1})\right]^2 + \left[(y_k - y_{k+1})/(t_k - t_{k+1})\right]^2}\ .$$

When the sign of the condition (9) changes, the process is locally adjusted with a minimum of costs, in contrast to the traditional approach, which assumes doubling the number of points on the entire segment $[A, B]$ to achieve the required accuracy. If the last adaptive step overlaps the boundary of B, then $f(B)$ is assumed to be a finite ordinate.

The value α in the formula (13), in addition to its main function, can also contain a tool putting the stepping process into oscillation mode, which would make it possible to better retain the proximity $\varepsilon \sim \left|y_k^* - g_k^*\right|$ if the function $f(x)$ abruptly changes (right up to the discontinuity of it first derivative) which can be done by adapting the coefficient α in the form

$$\alpha_k = \alpha_{k-1} \exp[\alpha_0(\varepsilon - \left|y_k^* - g_k^*\right|)], \quad \alpha_0 > 0.$$

3. Adaptive approach in the problem of calculating a definite integral

When the representation of a definite integral is not known, for example, in the form of an infinite convergent series, and it is impossible to ensure a given absolute accuracy by taking a certain number of terms of the series, it is necessary to directly use either the Newton-Cotes or Gauss formulas [2]. The former are more convenient and simple but inferior in accuracy to the latter at equal number of grid points.

The process of perfecting quadrature formulas since the moment of their appearance to our days has not ceased [18, 19]. However, if at the initial stage this was related to the choice of optimal points or quadrature coefficients, lately the problem of improvement has been translated into the category of adapting the parameters of one or another quadrature to the singularities of the integrand [2]. In a number of cases, the second stage, due to the use of adaptive control, leads to the results that are at the level of the optimal Gauss quadratures, and sometimes even superior to them [21–25]. The complexities of implementation that arise when solving such problems are primarily related to the appearance of difficult-to-compute functions introduced by the adopted models.

It is proposed to use a posteriori adaptive approach to numerical integration, especially since it is revealed that it is almost completely similar to the length of the curve (11). Indeed, an approximate representation of the definite integral (see Figure 3), with allowance for (13) and (14), can be expressed as the sum

$$S = 0.5\sum_{k=0}^{n-1} h_k \left(y_k^* + g_k^* \right).$$

All formulas of the type (13) providing an adaptive process remain in effect, the grid adjustments (14) completely coinciding. In other words, the algorithm allows us to obtain the values of both L and S, as well as a set of approximations $\{x_k^*, f_k^*\}$ on the segment $[A, B]$.

To guarantee the numerical stability of integration if the function $f(x)$ changes abruptly, it is possible to restrict the value of the step parameter (13) from below and from above.

4. Numerical experiment

The evaluation of the quality of a method is usually carried out according to various criteria [1, 2], which, in addition to its specific purpose, can also take into account a certain degree of universality in solving problems of a relatively wide area.

In evaluating the effectiveness of a method, we use a widely used approach that takes into account two most important characteristics: the achieved accuracy of the solution and the amount of computational costs N. Following [1], such a conditional criterion (efficiency index) in our approximation problem can, for example, be taken in the form

$$E = \frac{\ln\left(\left|S^*\right| + h_0^2\right) - \ln\left(\left|S - S^*\right| + h_0^2\right)}{N}, \qquad (15)$$

where the proximity measure $|S–S^*|\le\varepsilon$ on $[A, B]$ (see (13), (14) and fig. 2); S is an approximate solution obtained by a particular method, and S^* is the exact solution.

The table below presents the solutions to the problems with mainly difficult-to-compute functions proposed by well-known authors [7, 22–24, 26], in which the results of testing by both the trapezoidal method (TM) and adaptation method (AM) are compared with the values of the quality criterion E (15) and the number N of computations of $f(x)$ to achieve the accuracy ε. In this case the AECM accepts ε=0.01, α=10, and the common start is accepted as h_0=0.0625. In the table, $\sigma(x)$ stands for the unitary Heaviside jump function.

Calculation results of test cases

Function	$[A, B]$	Method	S	N	E
$y = 13(x - x^2)e^{-3x/2}$ [22]	[0; 4]	TM	–1.5529004234	65	0.081016
		AM	–1.5535812508	42	0.123441
$y = 1/(1 + (5x)^2)$ [7]	[–1; 4]	TM	0.5788232767	81	0.061715
		AM	0.5782095842	45	0.107864
$y = (x^2)^{1/3} - x$	[–1; 2]	TM	1.0022152690	49	0.102729
		AM	1.0066125146	31	0.167322
$y = [(x-0.3)^2 + 0.1^2]^{-1} + [(x-0.9)^2 + 0.2^2]^{-1} - 6$ [23]	[0; 1]	TM	29.8469834899	35	0.216577
		AM	29.8685381846	30	0.255216
$y = f_1\sigma(2-x) + \sigma(x-2)[f_2\sigma(5-x) + f_3\sigma(x-5)]$ $f_1 = \left\|(x-1)^4 - 1\right\|$, $f_2 = \left\|(x-3)^4 - 1\right\|$, $f_3 = \left\|(x-5)^4 + 15\right\|$	[–0.5; 6.5]	TM	30.0541000366	113	0.064510
		AM	30.0356075719	130	0.065790
$y = \left\|\cos x / \ln x\right\|$	[1.05; 8.5]	TM	4.0401378301	120	0.033874
		AM	3.9911147636	61	0.087370
$y = \left\|\sin x\right\| + \left\|\ln x\right\|$	[0.1; 6.6]	TM	11.5708938063	105	0.072655
		AM	11.5710118748	61	0.124723
$y = e^{x^2}$ (Dawson's integral) [26]	[0; 6.5]	TM	$1.762337 \cdot 10^{17}$	209	0.020629
		AM	$1.737279 \cdot 10^{17}$	193	0.035823
$y = \arcsin(\sin x) - x^{1/2}$	[0; 8]	TM	–13.6297344023	129	0.059185
		AM	–13.6328221601	50	0.162118
$y = x + 1 + (x-1)\sigma(x-1)(\sigma(3-x)+1)$ [24]	[0; 5]	TM	7.4375000000	81	0.058363
		AM	7.5461485167	45	0.111335

As you can see, the efficiency E of the AM proves to be higher than the uniform TM under the same initial conditions.

Conclusion

A simple mechanism of adaptive control of the stepwise process of piecewise linear approximation of a wide class of continuous functions with a finite number of discontinuities of the first and second kind on the segment [A, B] is given. This method illustrates a sufficiently high efficiency, using the exponential law of adaptation, which reacts quickly to the level of error deviation from the permissible value and allows obtaining more general results that are valid for a wide class of functions.

The cases considered, mainly devoted to the calculation of definite integrals of difficult-to-compute functions, have shown a rather high efficiency of the approach, owing to the use of adaptive control over the process of computations. For the operation of the method, knowledge of certain analytical properties of the function $f(x)$, for example, the existence of derivatives, the presence of characteristic points, etc., is not required, and a separate preliminary and comprehensive analysis of the function in question is excluded. The initial data are minimal: an integrand function $f(x)$ defined on an arbitrarily large finite domain [A, B], the required admissible accuracy ε of the deviation from $f(x)$, and an initial step.

The proposed self-adapting method guarantees the required accuracy in automatic mode. There is no need in choosing a suitable quadrature, a predefined special distribution of points and corresponding weight constants, and therefore, recomputation in case of their unsuccessful selection. The method is constructively simple, quite competitive among similar ones, based on a piecewise linear approximation of various functions. The result is implemented in a single pass without any preliminary transformations and solutions of equation systems, use of tables of quadrature coefficients and points. The scheme does not require that the information about the function $f(x)$ be stored simultaneously on the entire domain [A, B]. Only a small amount of RAM is involved and only direct computations are made. In situations with maximum a priori uncertainty, the workability and numerical stability of the method is maintained.

The reliability of the obtained results is confirmed by the solution of known test examples and their comparison with the results of solutions by other methods.

The proposed method opens the way for creating effective means of approximating functions and associated with them numerical integration differentiation, solutions of both finite, integral, and differential equations, summation of series, and so on.

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Received 16 March 2018