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DETERMINATION OF THE EQUI-STRESS HOLE SHAPE FOR A STRINGER PLATE WEAKENED BY A SURFACE CRACK

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On the basis of the principle of equal stress, a solution is given to the inverse problem of determining the optimal shape of the hole contour for a plate weakened by a surface rectilinear crack. The plate is reinforced by a regular system of elastic stiffeners (stringers). The crack originates from the hole contour perpendicular to the riveted stringers. The plate is subjected to uniform tension at infinity along the stiffeners. The plate under consideration is assumed to be either elastic or elastic-plastic. The criterion that determines the optimal shape of the hole is the condition that there is no stress concentration on the hole surface and the requirement that the stress intensity factor in the vicinity of the crack tip be equal to zero. In the case of an elastic-plastic plate, the plastic region at the moment of nucleation should encompass the entire hole contour at once, without deep penetration. The problem posed is to determine the hole shape at which the tangential normal stress acting on the contour is constant, and the stress intensity factor in the vicinity of the crack tip is zero, as well as to determine the magnitudes of the concentrated forces that replace both the action of the stringers and the stress-strain state of the reinforced plate. The method of a small parameter, the theory of analytic functions, and the method for direct solution of singular integral equations were used. The problem posed is reduced to the problem of finding a conditional extremum. The method of Lagrange indefinite multipliers was used. The obtained solution to the inverse problem allows increasing the bearing capacity of the stringer plate.

Keywords: *plate, stringers, equal strength hole, crack.*

Introduction

Thin plates (panels) as elements of various designs and machines are often weakened by technological holes. Finding an equi-stress hole contour is very important to prevent plate (panel) fracture and, accordingly, for the reliability of the structure or machine. Problems of finding an equi-stress hole contour were investigated in [1–20] and others, but cracks can always be present in a real material. There are only a few publications [21–25] on finding the optimal hole contour at which cracks will not grow in the material. The problem posed in this paper is to find the equi-stress hole contour in a plate, reinforced by a system of stringers and weakened by a rectilinear surface crack. A contour is searched at which there is no stress concentration near the hole and the crack is not growing.

Problem Formulation

Consider an unbounded thin plate (panel) reinforced by a regular system of stiffeners (stringers). The reinforced panel is subjected to a uniform tensile stress of $\sigma_y^\infty = \sigma_0$ along the stringers at infinity. The plate has a hole from which a rectilinear crack emanates (Fig. 1). The elastic isotropic stringers are considered to be riveted to the plate symmetrically relative to its surface in a discrete manner with a constant step of y_0 along the entire stringer length. The plate material is assumed to be isotropic.

The loading conditions are considered to be quasi-static. The following assumptions are made: during deformation the stringer thickness is unchanged, and the stress state is uniaxial; a flat stress state is realized in the plate. The stringers work only in tension (do not bend); the stringer system is a truss, and no stringer loosening occurs due to the setting of attachment points. The plate and stringers interact with each other in the same plane and only at the attachment points. The attachment points (adhesion areas) are the same, with a radius of a_0, which is small compared to their step of $2L$ and other typical dimensions.

The action of the attachment points is replaced by the action of the concentrated forces applied at the points corresponding to the centers of the adhesion areas: $z=\pm(2m+1)L\pm iky_0$ (m=0, 1, 2, …; k=1, 2, …). The action of the stringers is replaced by the equivalent concentrated forces P_{mn} applied at the points of their connection with the plate. These forces are unknown in advance.

The boundary conditions of the problem have the form

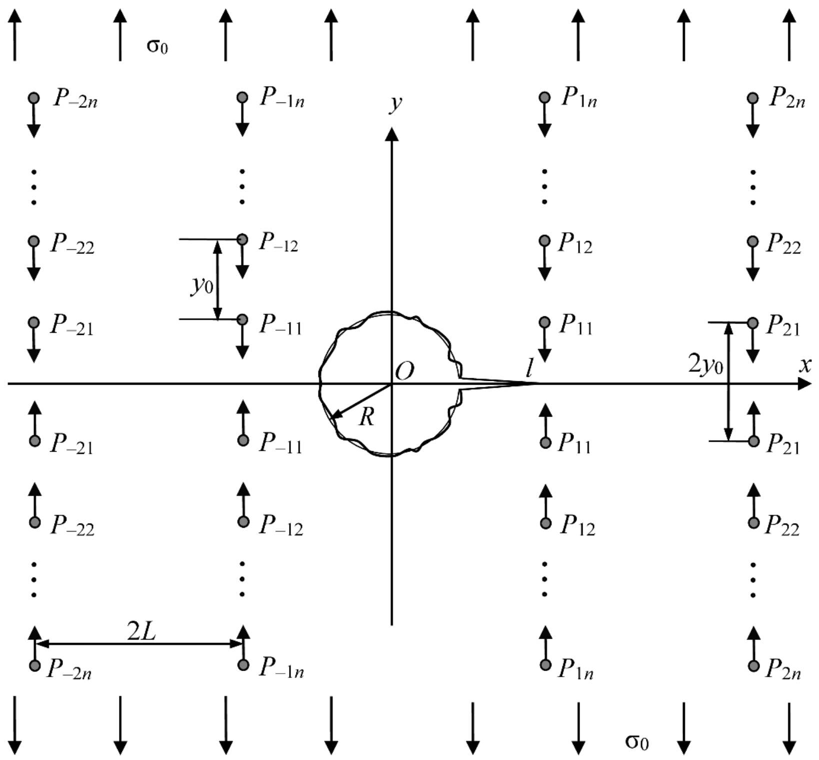

Fig. 1. Design scheme of the problem of finding an equi-stress hole contour

– along the hole contour

$$\sigma_n = 0, \qquad \tau_{nt} = 0;$$

– on the crack faces

$$\sigma_y = 0, \qquad \tau_{xy} = 0 \qquad R \le x \le l.$$

Here, n and t are the normal and tangent to the hole contour.

The task is to find the shape of the hole (the unknown contour L_0) that satisfies two conditions: the tangential normal stress σ_t acting on the contour is constant, i.e.

$$\sigma_t = \sigma_* = \text{const} \quad \text{along the hole contour;} \tag{1}$$

and there is no growth of the crack emanating from the hole. Since, according to the theory of quasi-brittle fracture by Irwin-Orowan, the stress intensity factor is taken as the parameter characterizing the stress state in the vicinity of the crack tip, this condition means the requirement that the stress intensity factor K_{I}^l in the vicinity of the crack tip be zero

$$K_{\mathrm{I}}^l = 0. \tag{2}$$

In condition (1), the quantity σ_* requires that it be determined if the plate material is elastic. For an elastic-plastic plate, we require that the plastic region at the moment of crack nucleation encompass the entire hole contour at once, without deep penetration, since such a body is the most durable in the sense of uniform stress distribution over the entire hole contour [1]. We write the plasticity condition as follows [26]:

$$f(\sigma_n, \sigma_t, \tau_{nt}) = 0,$$

where f is a given function. It follows from the plasticity condition that for an elastic-plastic plate $\sigma_* = \sigma_s$, i.e. condition (1) has the form

$$\sigma_t = \sigma_s = \text{const} \quad \text{along the hole contour,}$$

where σ_s is the plasticity constant of the plate material.

Solution of the boundary-value problem

We will look for the unknown hole contour L_0 in the class of contours close to circular. We represent the required contour L_0 as

$$r=\rho(\theta)=R+\varepsilon H(\theta).$$

Here, R is the radius of a circular hole; $\varepsilon=R_{\max}/R$ is a small parameter; $R_{\max}$ is the maximum height of the protrusion (depression) of the unevenness of the profile of the hole contour L_0 from the circle $r=R$.

The function $H(\theta)$ is required to be determined in the process of solving the problem. Without limiting the generality of the inverse problem under consideration, we assume that the function $H(\theta)$ is symmetric with respect to the coordinate axes and can be represented as a Fourier series $H(\theta)=\sum_{k=1}^{\infty} d_{2k}\cos 2k\theta$.

The required functions, i.e. stresses, displacements, the concentrated forces P_{mn} and the stress intensity factor K_{I}^{l} are sought in the form of expansions in powers of the small parameter ε

$$\sigma_n=\sigma_n^{(0)}+\varepsilon\sigma_n^{(1)}+\ldots,\qquad \sigma_t=\sigma_t^{(0)}+\varepsilon\sigma_t^{(1)}+\ldots,\qquad \tau_{nt}=\tau_{nt}^{(0)}+\varepsilon\tau_{nt}^{(1)}+\ldots,$$

$$u_n=u_n^{(0)}+\varepsilon u_n^{(1)}+\ldots,\qquad v_n=v_n^{(0)}+\varepsilon v_n^{(1)}+\ldots,$$

$$P_{mn}=P_{mn}^{(0)}+\varepsilon P_{mn}^{(1)}+\ldots,$$

$$K_{\mathrm{I}}=K_{\mathrm{I}}^{(0)}+\varepsilon K_{\mathrm{I}}^{(1)}+\ldots.$$

Each of these approximations satisfies the system of differential equations of the plane problem of the theory of elasticity. For the sake of simplicity, we neglect the terms containing ε of order higher than one.

Expanding the expressions for the stresses in the vicinity of $r=R$ in a series, we obtain the values of the stress tensor components at $r=\rho(\theta)$. Using the well-known formulas [27] for the stress components σ_n and τ_{nt}, we write the boundary conditions of the problem

– for the zeroth approximation

$$\sigma_r^{(0)}=0,\qquad \tau_{r\theta}^{(0)}=0\quad \text{along the contour } r=R, \tag{3}$$

$$\sigma_x^{(0)}=0,\qquad \tau_{xy}^{(0)}=0\qquad R\le x\le l\quad \text{on the crack faces;} \tag{4}$$

– for the first approximation

$$\sigma_r^{(1)}=N,\qquad \tau_{r\theta}^{(1)}=T\quad \text{along the contour } r=R,$$

$$\sigma_x^{(1)}=0,\qquad \tau_{xy}^{(1)}=0\qquad R\le x\le l\quad \text{on the crack faces,}$$

$$N=-H(\theta)\frac{\partial\sigma_r^{(0)}}{\partial r}+2\frac{\tau_{r\theta}^{(0)}}{R}\frac{\partial H(\theta)}{\partial\theta},\qquad T=-H(\theta)\frac{\partial\tau_{r\theta}^{(0)}}{\partial r}+\frac{\sigma_\theta^{(0)}-\sigma_r^{(0)}}{R}\frac{\partial H(\theta)}{\partial\theta}.$$

Based on the Kolosov-Muskhelishvili formulas [27] and boundary conditions (3) along the hole contour and boundary conditions (4) on the crack faces, the problem is reduced in the zeroth approximation to the determination of two analytic functions $\Phi^{(0)}(z)$, $\Psi^{(0)}(z)$ from the conditions

$$\Phi^{(0)}(\tau)+\overline{\Phi^{(0)}(\tau)}-e^{2i\theta}\left[\overline{\tau}\Phi^{(0)\prime}(\tau)+\Psi^{(0)}(\tau)\right]=0\qquad \text{at } \tau=Re^{i\theta}, \tag{5}$$

$$\Phi^{(0)}(x)+\overline{\Phi^{(0)}(x)}+x\overline{\Phi^{(0)\prime}(x)}+\overline{\Psi^{(0)}(x)}=0\qquad R\le x\le l. \tag{6}$$

The solution to the boundary-value problem (5)–(6) is sought in the form

$$\Phi^{(k)}(z)=\Phi_0^{(k)}(z)+\Phi_1^{(k)}(z)+\Phi_2^{(k)}(z), \tag{7}$$

$$\Psi^{(k)}(z)=\Psi_0^{(k)}(z)+\Psi_1^{(k)}(z)+\Psi_2^{(k)}(z),$$

where $k=0$.

The potentials $\Phi_0^{(0)}(z)$ and $\Psi_0^{(0)}(z)$ describe the stress and strain field in a solid (without a crack) plate under the action of a system of the concentrated forces $P_{mn}^{(0)}$ and the stress σ_0. $\Phi_0^{(0)}(z)$, $\Psi_0^{(0)}(z)$ are determined by the formulas

$$\Phi_0^{(0)}(z)=\frac{1}{4}\sigma_0-\frac{i}{2\pi h(1+\kappa)}\sum_{m,n}{}' P_{mn}^{(0)}\left[\frac{1}{z-mL+iny_0}-\frac{1}{z-mL-iny_0}\right], \tag{8}$$

$$\Psi_0^{(0)}(z)=\frac{1}{2}\sigma_0-\frac{i\kappa}{2\pi h(1+\kappa)}\sum_{m,n}{}' P_{mn}^{(0)}\left[\frac{1}{z-mL+iny_0}-\frac{1}{z-mL-iny_0}\right]+$$

$$+\frac{i}{2\pi h(1+\kappa)}\sum_{m,n}{}' P_{mn}^{(0)}\left[\frac{mL-iny_0}{(z-mL+iny_0)^2}-\frac{mL+iny_0}{(z-mL-iny_0)^2}\right],$$

where κ=(3– ν)/(1+ν); ν is the Poisson ratio of the plate material; the prime at the summation symbol indicates that the index *m*=*n*=0 is excluded during the summation.

The potentials $\Phi_1^{(0)}(z)$ and $\Psi_1^{(0)}(z)$ are sought in the form

$$\Phi_1^{(0)}(z)=\frac{1}{2\pi}\int_R^l\frac{g^{(0)}(z)}{t-z}dt, \tag{9}$$

$$\Psi_1^{(0)}(z)=\frac{1}{2\pi}\int_R^l\left[\frac{1}{t-z}-\frac{1}{(t-z)^2}\right]g^{(0)}(z)dt,$$

where $g^{(0)}(x)=\frac{2\mu}{1+\kappa}\frac{d}{dx}\left[v^+(x,0)-v^-(x,0)\right]$; μ is the shear modulus of the plate material.

The required function $g^{(0)}(x)$ and the potentials $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$ must be determined from conditions (5)–(6). We represent the boundary condition (5) in the form

$$\Phi_2^{(0)}(\tau)+\overline{\Phi_2^{(0)}(\tau)}-e^{2i\theta}\left[\bar\tau\Phi_2^{(0)\prime}(\tau)+\Psi_2^{(0)}(\tau)\right]=-\Phi_*^{(0)}(\tau)-\overline{\Phi_*^{(0)}(\tau)}+e^{2i\theta}\left[\bar\tau\Phi_*^{(0)\prime}(\tau)+\Psi_*^{(0)}(\tau)\right], \tag{10}$$

where $\Phi_*^{(0)}(\tau)=\Phi_0^{(0)}(\tau)+\Phi_1^{(0)}(\tau)$, $\Psi_*^{(0)}(\tau)=\Psi_0^{(0)}(\tau)+\Psi_1^{(0)}(\tau)$.

Solving the boundary-value problem (10) with the N. I. Muskhelishvili method [27], we determine the potentials $\Phi_2^{(0)}(z)$ and $\Psi_2^{(0)}(z)$

$$\Phi_2^{(0)}(z)=\frac{\sigma_0}{2z^2}+\frac{1}{2\pi}\int_R^l\left[\frac{1-t^2}{t(1-tz)}+\frac{z-t}{(1-tz)^2}\right]g^{(0)}(t)dt-\frac{i}{2\pi h(1+\kappa)}\times$$

$$\times\sum_{m,n}{}' P_{mn}^{(0)}\left\{\frac{(mL-iny_0)(mL+iny_0)-1}{(mL-iny_0)\left[z(mL-iny_0)-1\right]^2}-\frac{(mL+iny_0)(mL-iny_0)-1}{(mL+iny_0)\left[z(mL+iny_0)-1\right]^2}\right\}+$$

$$+\frac{i\kappa}{2\pi h(1+\kappa)}\sum_{m,n}{}' P_{mn}^{(0)}\left\{\frac{1}{z\left[z(mL-iny_0)-1\right]}-\frac{1}{z\left[z(mL+iny_0)-1\right]}\right\}; \tag{11}$$

$$\Psi_2^{(0)}(z)=\frac{\sigma_0}{2z^2}+\frac{\Phi_2^{(0)}(z)}{z^2}-\frac{\Phi_2^{(0)\prime}(z)}{z^2}+$$

$$+\frac{1}{2\pi z}\int_R^l\left[\frac{2}{tz}-\frac{t}{z(1-tz)}+\frac{t^2z-z-t}{z(1-tz)^2}-\frac{2t(z-t)}{(1-tz)^3}\right]g^{(0)}(t)dt+\frac{i}{2\pi h(1+\kappa)z}\times$$

$$\times\sum_{m,n}{}' P_{mn}^{(0)}\left\{\frac{1}{z(mL-iny_0)-1}-\frac{1}{z(mL+iny_0)-1}+\frac{1}{z(mL-iny_0)}-\frac{1}{(zmL+iny_0)}\right\}.$$

In formulas (11), all linear dimensions are referred to the radius R.

Requiring that functions (7) at k=0 satisfy the boundary condition (6) on the crack faces, we obtain a singular integral equation for $g^{(0)}(x)$

$$\frac{1}{\pi}\int_R^l \frac{g^{(0)}(t)}{t-x}dt+\frac{1}{\pi}\int_R^l K(t,x)g^{(0)}(t)dt=F^{(0)}(x)\,, \qquad (12)$$

$$F(x)=f_0^{(0)}(x)+f_1^{(0)}(x)\,,$$

$$f_0^{(0)}(x)=-\sigma_0+\frac{\kappa+2}{\pi h(1+\kappa)}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P_{-mn}^{(0)}ny_0\left[\frac{1}{(x-mL)^2+n^2y_0^2}\right]+\sum_{m,n=1}^{\infty}P_{mn}^{(0)}ny_0\left[\frac{1}{(x+mL)^2+n^2y_0^2}\right]\right\}-$$

$$-\frac{1}{\pi h(1+\kappa)}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P_{mn}^{(0)}ny_0\frac{(x-mL)^2-n^2y_0^2-(x^2-m^2L^2)}{\left[(x-mL)^2+n^2y_0^2\right]^2}+\right.$$

$$\left.+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}P_{-mn}^{(0)}ny_0\frac{(x+mL)^2-n^2y_0^2-(x^2-m^2L^2)}{\left[(x+mL)^2+n^2y_0^2\right]^2}\right\},$$

$$f_1^{(0)}(x)=-\frac{1}{2\pi(1+\kappa)h}\sum_{m,n=1}^{\infty}P_{-mn}^{(0)}n\left\{\left(2+\frac{1}{x^2}\right)\times\right.$$

$$\times\left\langle\frac{2(m^2L^2+n^2y_0^2-1)\left[x^2(3m^2L^2-n^2y_0^2)+4xmL+1\right]}{(m^2L^2+n^2y_0^2)\left[(xmL+1)^2+x^2n^2y_0^2\right]^2}+\frac{2\kappa}{(mxL+1)^2+x^2n^2y_0^2}\right\rangle+$$

$$+\left(\kappa-\frac{1}{\kappa}\right)\left\langle-4(m^2L^2+n^2y_0^2-1)\frac{\left[x^3(3m^2L^2-n^2y_0^2)+6x^2mL+3x\right]}{\left[(xmL+1)^2+x^2n^2y_0^2\right]^3}-\right.$$

$$\left.\left.-4\kappa\frac{mL+x(m^2L^2+n^2y_0^2)}{\left[(mxL+1)^2+x^2n^2y_0^2\right]^2}\right\rangle-2\left[\frac{1}{(mxL+1)^2+x^2n^2y_0^2}+\frac{1}{m^2L^2+n^2y_0^2}\right]\right\}-$$

$$-\frac{1}{2\pi(1+\kappa)h}\sum_{m,n=1}^{\infty}P_{mn}^{(0)}\left(2+\frac{1}{x^2}\right)\left\langle\frac{2(m^2L^2+n^2y_0^2-1)\left[x^2(3m^2L^2-n^2y_0^2)-4xmL+1\right]}{(m^2L^2+n^2y_0^2)\left[(xmL-1)^2+x^2n^2y_0^2\right]^2}+\right.$$

$$\left.+\frac{2\hat{e}_0}{(mxL-1)^2+x^2n^2y_0^2}\right\rangle+\left(\kappa-\frac{1}{\kappa}\right)\times$$

$$\times\left\langle-4(m^2L^2+n^2y_0^2-1)\frac{x^3(3m^2L^2-n^2y_0^2)-6x^2mL+3x}{\left[(xmL-1)^2+x^2n^2y_0^2\right]^3}+4\kappa\frac{mL-x(m^2L^2+n^2y_0^2)}{\left[(mxL-1)^2+x^2n^2y_0^2\right]^2}\right\rangle-$$

$$\left.-2\left[\frac{1}{(mxL-1)^2+x^2n^2y_0^2}+\frac{1}{m^2L^2+n^2y_0^2}\right]\right\}-\frac{\sigma_0}{2x^2}-\frac{3}{2}\frac{\sigma_0}{x^4}\,.$$

Using the method of direct solution of singular integral equations [28–30], we seek a solution to equation (12). Turning to dimensionless variables, we represent it in the form

$$g^{(0)}(\eta)=\frac{g_0^{(0)}(\eta)}{\sqrt{1-\eta^2}}\,,$$

where $g_0^{(0)}(\eta)$ is a bounded function, continuous on the interval [−1, 1]; it is replaced by the Lagrange interpolation polynomial constructed from the Chebyshev nodes.

In the problem under consideration, one end of the crack reaches the surface of the hole. The stresses at this end are limited. Together with the additional condition

$$K_{\mathrm{I}}^{(0)}=0 \qquad \text{at } x=R,$$

the singular integral equation (12), is reduced, through the algebraization procedure [28–30], to a system of M linear algebraic equations for determining the M unknowns $g^{(0)}(\tau_m)$ (m=1, 2, …, M)

$$\begin{cases}\sum\limits_{k=1}^{M} A_{mk} g_k^{(0)} = f_0^{(0)}(\eta_m)+f_1^{(0)}(\eta_m) \\ \sum\limits_{k=1}^{M}(-1)^{k+M} g_k^{(0)}\,\mathrm{tg}\dfrac{\theta_k}{2}=0\end{cases}, \tag{13}$$

where $A_{mk}=\dfrac{1}{M}\left[\dfrac{1}{\sin\theta_m}\mathrm{ctg}\dfrac{\theta_m+(-1)^{|m-k|}\theta_k}{2}+K_0(\eta_m,\tau_k)\right]$; m=1,2,…,M–1; $\theta_m=\dfrac{2m-1}{2M}\pi$; $\eta_m=\cos\theta_m$; $\tau_k=\eta_k$; $g_k^{(0)}=g^{(0)}(\tau_k)$.

Next, we determine the concentrated forces $P_{mn}^{(0)}$. According to Hooke's law, the magnitude of the concentrated force $P_{mn}^{(0)}$ acting on each attachment point from the side of the stringer will be

$$P_{mn}^{(0)}=\frac{E_s A_s}{2y_0 n}\Delta v_{mn}^{(0)} \qquad (m,\, n=1,2,\ldots), \tag{14}$$

where E_s is Young's modulus of the stringer material; A_s is the cross-section of the stringer; $2y_0n$ is the distance between the attachment points; $\Delta v_{mn}^{(0)}$ the mutual displacement of the attachment points under consideration, equal to the elongation of the corresponding stringer segment.

Let us accept [31] the natural assumption that the condition of compatibility of displacements is satisfied, that is, we assume that the mutual elastic displacement of the points $mL+i(ny_0-a_0)$ and $mL-i(ny_0-a_0)$ and the mutual displacement of the attachment points $\Delta v_{mn}^{(0)}$ in the problem under consideration are equal. Using the Kolosov-Muskhelishvili formulas [27] and relations (7)–(9), (11), we find the mutual displacement of the attachment points $\Delta v_{mn}^{(0)}$. Solving systems (13) and (14), we find the magnitudes of the concentrated forces $P_{mn}^{(0)}$, the approximate values of the function $g^{(0)}(\tau_m)$ at the nodal points, and thus the complex potentials of the zeroth approximation.

For the stress intensity factor in the vicinity of the crack tip in the zeroth approximation, we have

$$K_{\mathrm{I}}^{(0)}=\sqrt{\pi(l-R)}\sum_{m=1}^{M}(-1)^m g^{(0)}(t_m)\,\mathrm{ctg}\frac{2m-1}{4M}\pi .$$

Using the Kolosov-Muskhelishvili formulas and relations (7), we find the stress state in the stringer plate in the zeroth approximation. Knowing the stress components in the zeroth approximation, we find the functions N and T.

Next, a solution to the problem is constructed in the first approximation. The boundary conditions of the problem for the first approximation are written in the form

$$\Phi^{(1)}(\tau)+\overline{\Phi^{(1)}(\tau)}-e^{2i\theta}\left[\bar{\tau}\Phi^{(1)\prime}(\tau)+\Psi^{(1)}(\tau)\right]=N-iT\,, \tag{15}$$

$$\Phi^{(1)}(x)+\overline{\Phi^{(1)}(x)}+\overline{x\Phi^{(1)\prime}(x)}+\overline{\Psi^{(1)}(x)}=0 \qquad R\le x\le l. \tag{16}$$

Similarly to the zeroth approximation, we seek a solution to the boundary-value problem (15) in the form (7) at k=1, where the potentials $\Phi_0^{(1)}(z)$ and $\Psi_0^{(1)}(z)$ describe the stress and strain field under the action of a system of the concentrated forces $P_{mn}^{(1)}$ and are determined by formulas similar to (8), and we should set σ_0 equal to zero, and replace $P_{mn}^{(0)}$ by $P_{mn}^{(1)}$. The potentials $\Phi_1^{(1)}(z)$ and $\Psi_1^{(1)}(z)$ are sought in a form similar to (9), with the function $g^{(0)}(x)$ replaced by $g^{(1)}(x)$. The potentials $\Phi_2^{(1)}(z)$ and $\Psi_2^{(1)}(z)$ are determined from the boundary condition (15), again with the N. I. Muskhelishvili method:

$$\Phi_2^{(1)}(z)=\Phi_*^{(1)}(z)+\sum_{k=0}^{\infty}a_{2k}z^{-2k}, \qquad \Psi_2^{(1)}(z)=\Psi_*^{(1)}(z)+\sum_{k=0}^{\infty}b_{2k}z^{-2k}.$$

Here, $\Phi_*^{(1)}(z)$, $\Psi_*^{(1)}(z)$ are determined by formulas similar to (11), in which σ_0 should be set equal to zero, and $P_{mn}^{(0)}$ and $g^{(0)}(x)$ should be replaced by $P_{mn}^{(1)}$ and $g^{(1)}(x)$, respectively. The coefficients a_{2k} and b_{2k} are found by the formulas

$$a_{2n}=C_{2n}R^{2n} \quad (n=1, 2,\ldots), \qquad a_0=0,$$
$$b_{2n}=(2n-1)R^2a_{2n-2}-R^{2n}a_{-2n+2} \quad (n\geq 2),$$
$$b_0=0, \qquad b_2=-C_0R^2, \qquad N-iT=\sum_{k=-\infty}^{\infty}C_{2k}e^{-2ki\theta}.$$

For the concentrated forces $P_{mn}^{(1)}$, we have

$$P_{mn}^{(1)}=\frac{E_sA_s}{2y_0n}\Delta v_{mn}^{(1)}. \tag{17}$$

The mutual displacement of the attachment points $\Delta v_{mn}^{(1)}$ is determined in the same way as the zeroth approximation.

Requiring that functions (7) for k=1 satisfy condition (16) on the crack faces in the first approximation, we obtain a singular integral equation for the function $g^{(1)}(x)$

$$\frac{1}{\pi}\int_R^l\frac{g^{(1)}(t)}{t-x}dt+\frac{1}{\pi}\int_R^l K(t,x)g^{(1)}(t)dt=F^{(1)}(x). \tag{18}$$

Using the algebraization procedure [28–30], we reduce the singular integral equation (18) under the condition

$$K_{\mathrm{I}}^{(1)}=0 \text{ at the hole edge,}$$

similarly to the zeroth approximation, to a system of M linear algebraic equations for determining the M unknowns $g^{(1)}(\tau_m)$ (m=1, 2,…, M)

$$\begin{cases}\sum_{k=1}^{M}A_{mk}g_k^{(1)}=f_0^{(1)}(\eta_m)+f_1^{(1)}(\eta_m)\\ \sum_{k=1}^{M}(-1)^{k+M}g_k^{(1)}\,\mathrm{tg}\frac{\theta_k}{2}=0\end{cases}, \tag{19}$$

where m=1, 2,…, M–1; $g_k^{(1)}=g^{(1)}(\tau_k)$.

For the stress intensity factor in the vicinity of the crack tip in the first approximation, we have

$$K_{\mathrm{I}}^{(1)}=\sqrt{\pi(l-R)}\sum_{m=1}^{M}(-1)^{m+M}g^{(1)}(t_m)\,\mathrm{ctg}\frac{2m-1}{4M}\pi.$$

The right-hand sides of the obtained systems of equations of the first approximation contain the coefficients d_{2k} of the expansion of the function $H(\theta)$ in a Fourier series. Thus, for the obtained systems to be closed, the missing equations should be constructed using conditions (1) and (2). Using the obtained solution, we find σ_t in the surface layer of the contour L_0 (r=$\rho(\theta)$) up to the first- order values with respect to the small parameter ε

$$\sigma_t=\sigma_t^{(0)}(\theta)\Big|_{r=R}+\varepsilon\left[\frac{H(\theta)\partial\sigma_t^{(0)}(\theta)}{\partial r}+\sigma_t^{(1)}(\theta)\right]_{r=R}.$$

The stresses $\sigma_t^{(1)}(\theta)$ depend on the coefficients d_{2k} of the Fourier series of the required function $H(\theta)$. Requiring that the close-to-uniform stress distribution be ensured along the hole contour, we obtain the missing equations that allow determining the coefficients d_{2k}.

The stress concentration along the hole contour is reduced by minimizing the criterion

$$U = \sum_{i=1}^{M} [\sigma_t(\theta_i) - \sigma_*]^2 \to \min .$$

Here, σ_* is the unknown optimal value of the normal tangential stress in the surface layer of the hole for the elastic plate.

It is necessary to find the values of the unknown coefficients d_{2k} that provide the best possible values of the function $\sigma_t(\theta_i)$ according to condition (1) under the additional constraints (2). The function U and the stress intensity factor depend on the coefficients d_{2k}, and, thus, we arrive at the problem for the conditional extremum of the function $U_0(\sigma_*, d_{2k})$, when the coefficients d_{2k} are related to the additional constraints (2).

It is necessary to find the minimum value of the function $U_0(\sigma_*, d_{2k})$, and with k+1 arguments of this function being dependent and subject to the additional constraints (2).

To solve the problem for a conditional extremum, we use the method of Lagrange indefinite multipliers. Consider the auxiliary function

$$U_0 = U + \lambda K_1^l$$

with an indefinite factor λ.

The k+1 necessary conditions for the extremum have the form

$$\frac{\partial U_0}{\partial d_{2k}} = 0 \qquad (k=1,\, 2,\, \ldots,\, n), \qquad \frac{\partial U_0}{\partial \sigma_*} = 0 . \tag{20}$$

The obtained n+1 equations with the additional equation (2) constitute a system of equations with the n+1+1 unknowns σ_*, d_{2k} (k=1, 2,…, n), λ. Adding this system of equations to the previously obtained algebraic systems (17), (19), we obtain a closed algebraic system for determining all unknowns, including σ_* and the coefficients d_{2k}.

The system of equations (20), together with the obtained algebraic systems of the problem of the theory of elasticity in the zeroth and first approximations, makes it possible to determine the shape of an equi-stress hole, the stress-strain state of the reinforced plate, and in the case of an elastic plate also the optimal value of the normal tangential stress σ_*.

The calculations were carried out for the values of free parameters: a_0/L=0.01; y_0/L=0.25. It was assumed that E=7.1·10^4 MPa, E_s=11.5·10^4 MPa, stringers were made of the Al-steel composite, and the plate was made of the B95 alloy. For simplicity, it was assumed: A_s/y_0h=1. The number of stringers and attachment points was assumed to be 14, and the value M=72. The results of calculating the expansion coefficients of the required function $H(\theta)$ are given below.

d_2	d_4	d_6	d_8	d_{10}	d_{12}	d_{14}
0.1307	-0.1041	0.0773	0.0594	-0.0365	0.0139	0.0108

Conclusions

A solution has been found to the problem of finding an equi-stress hole contour in a plate reinforced by a regular stringer system. The found contour ensures the stability of a rectilinear surface crack emanating from the hole and the absence of stress concentration near the hole. A closed system of algebraic equations has been constructed for the found equi-stress hole shape. The obtained solution to the inverse problem makes it possible to increase the strength of the plate, as well as its reliability and bearing capacity.

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Визначення форми рівноміцного отвору для стрингерної пластини, ослабленої поверхневою тріщиною

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На основі принципу рівноміцності дається розв'язок оберненої задачі з визначення оптимальної форми контура отвору для пластини, ослабленої поверхневою прямолінійною тріщиною. Пластина підкріплена регулярною системою пружних ребер жорсткості (стрингерів). Тріщина виходить з контура отвору перпендикулярно приклепаним стрингерам. Пластина піддається на нескінченності однорідному розтягуванню уздовж ребер жорсткості. Пластина, що розглядається, припускається пружною або пружно-пластичною. Критерієм, що визначає оптимальну форму отвору, служить умова відсутності концентрації напруження на поверхні отвору і вимога рівності нулю коефіцієнта інтенсивності напружень в околі вершини тріщини. У разі пружно-пластичної пластини пластична область у момент зародження має охоплювати відразу увесь контур отвору, не проникаючи вглиб. Поставлена задача полягає у визначенні такої форми отвору, за якої тангенціальне нормальне напруження, що діє на контурі, є сталим, а коефіцієнт інтенсивності напруження в околі вершини тріщини дорівнює нулю, а також у визначенні величин зосереджених сил, що замінюють дію стрингерів, і напружено-деформованого стану підкріпленої пластини. Використовувалися метод малого параметра, теорія аналітичних функцій і метод прямого розв'язання сингулярних інтегральних рівнянь. Поставлена задача зводиться до задачі з відшукування умовного екстремуму. Застосовувався метод невизначених множників Лагранжа. Отриманий розв'язок оберненої задачі дозволяє підвищити несучу здатність пластини стрингера.

***Ключові слова**: пластина, стрингери, рівноміцний отвір, тріщина.*

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