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ANALYSIS OF THE STRESS STATE OF A LAYER WITH TWO CYLINDRICAL ELASTIC INCLUSIONS AND MIXED BOUNDARY CONDITIONS

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The spatial problem of the elasticity theory is studied for a layer with two infinite circular solid cylindrical inclusions that are parallel to each other and to the layer boundaries. The physical characteristics of the layer and the inclusions are different from each other and they are homogeneous, isotropic materials. The spatial function of stresses is given at the upper boundary, and the function of displacements is given at the lower layer boundary. Circular cylindrical elastic inclusions are rigidly connected to the layer. It is necessary to determine the stress-strain state of the composite body. The problem solution is based on the generalized Fourier method, which uses special formulas for the transition between the basic solutions of the Lamé equation in different coordinate systems. Thus, the layer is considered in the Cartesian coordinate system, the inclusions – in the local cylindrical ones. Satisfying the boundary and conjugation conditions, systems of infinite integro-algebraic equations were obtained, which were subsequently reduced to linear algebraic ones. The resulting infinite system is solved by the reduction method. After determining the unknowns, it is possible to find the stress values at any point of the elastic composite body. In numerical studies, a comparative analysis of the stress state in the layer and on the surfaces of inclusions at different distances between them is carried out. The analysis showed that when the inclusions approach each other, the stress state in the layer practically does not change. However, its significant change is observed in the bodies of inclusions, so with dense reinforcement ($(R_1 + R_2) / L > 0.5$), it is necessary to take into account the distances between the reinforcing fibers. At stress values from 0 to 1 and the order of the system of equations $m=10$, the accuracy of meeting the boundary conditions was 10^{-4}. With an increase in the system order, the accuracy of meeting the boundary conditions will increase. The given analytical-numerical solution can be used for high-precision determination of the stress-strain state of the given type of problems, and also as a reference for problems based on numerical methods.

Keywords: *composite, cylindrical inclusions in a layer, generalized Fourier method.*

Introduction

In various industries, one has to face the design of parts and structures from composite materials, which often are a layer with longitudinal reinforcement, where, based on engineering calculations, it is necessary to place reinforcement elements at a close distance to each other. In view of this, it is important to understand how the distance between the longitudinal rods affects the stress-strain state in the zone of contact between the layer and the reinforcement. To solve these problems, it is possible to conduct experiments [1, 2] or use a numerical-experimental approach [3–7], perceiving the composite as a single layer. Another approach is numerical-analytical, in which the composite body is perceived as a component of the intersection of several elements. However, this requires an efficient and highly accurate method of the stress-strain state calculation. Thus, in paper [8] the two-dimensional boundary value problem of diffraction of symmetric normal longitudinal shear waves for a layer with a cylindrical cavity or inclusion was solved using the method of images, and in papers [9–12], stationary wave diffraction problems of stress determination for a layer with a cylindrical cavity or inclusion were solved on the basis of the Fourier series decomposition method.

However, for spatial models with a large number of boundary surfaces and high accuracy of determining the stress state, the numerical-analytical generalized Fourier method [13] is most suitable. On the basis of this method, the problems for a half-space with a cavity or an inclusion [14–16], a layer with a cavity or inclusions [17–20], a cylinder with cylindrical cavities or inclusions [21–24], as well as for a half-space with a cavity coupled to a layer [25] are considered.

The generalized Fourier method is also used in the paper during research studies.

Problem statement

In an elastic homogeneous layer, there are two cylindrical inclusions with radii R_p made of materials different from the layer material and placed parallel to its boundaries.

The inclusions are considered in local cylindrical coordinate systems (ρ_p, φ_p, z), the layer – in the Cartesian coordinate system (x, y, z), combined with the coordinate system of the inclusion with the number p=1. The boundaries of the layer are located at a distance of $y=h$ and $y=-\tilde{h}$ (Fig. 1).

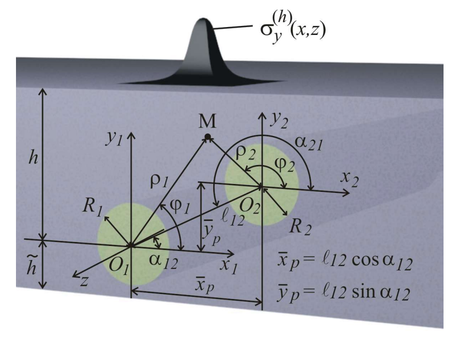


Fig. 1. A layer with two cylindrical inclusions

It is necessary to find a solution to the Lamé equation, provided that stresses are given at the upper layer boundary $F\vec{U}_0(x,z)\big|_{y=h} = \vec{F}_h^0(x,z)$, displacements are given on the lower one $\vec{U}_0(x,z)\big|_{y=-\tilde{h}} = \vec{U}_{\tilde{h}}^0(x,z)$, conjugation conditions are given on the boundaries of the contact of the layer and inclusions

$$\vec{U}_0(\varphi_p, z)\Big|_{\rho_p=R_p} = \vec{U}_p(\varphi_p, z)\Big|_{\rho_p=R_p}, \tag{1}$$

$$F\vec{U}_0(\varphi_p, z)\Big|_{\rho_p=R_p} = F\vec{U}_p(\varphi_p, z)\Big|_{\rho_p=R_p}, \tag{2}$$

where $\vec{U}_0$ is displacement in the layer; $\vec{U}_p$ is displacement in the cylindrical inclusion; $F\vec{U} = 2\cdot G\cdot\left[\dfrac{\sigma}{1-2\cdot\sigma}\vec{n}\cdot\operatorname{div}\vec{U} + \dfrac{\partial}{\partial n}\vec{U} + \dfrac{1}{2}\left(\vec{n}\times\operatorname{rot}\vec{U}\right)\right]$ is stress operator;

$$\begin{aligned}\vec{F}_h^0(x,z) &= \tau_{yx}^{(h)}\vec{e}_x + \sigma_y^{(h)}\vec{e}_y + \tau_{yz}^{(h)}\vec{e}_z\\ \vec{U}_{\tilde{h}}^0(x,z) &= U_x^{(\tilde{h})}\vec{e}_x + U_y^{(\tilde{h})}\vec{e}_y + U_z^{(\tilde{h})}\vec{e}_z\end{aligned} \tag{3}$$

known functions, which are considered as rapidly decreasing from the origin along the axis z and x.

Solution method

The basic solutions of the Lamé equation for Cartesian and cylindrical coordinate systems are chosen in the form [11]:

$$\begin{aligned}&\vec{u}_k^{\pm}(x,y,z;\lambda,\mu) = N_k^{(d)} e^{i(\lambda z+\mu x)\pm\gamma y};\\ &\vec{R}_{k,m}(\rho,\varphi,z;\lambda) = N_k^{(p)} I_m(\lambda\rho) e^{i(\lambda z+m\varphi)};\\ &\vec{S}_{k,m}(\rho,\varphi,z;\lambda) = N_k^{(p)}\left[(\operatorname{sign}\lambda)^m K_m(|\lambda|\rho)\cdot e^{i(\lambda z+m\varphi)}\right], k=1,2,3;\end{aligned} \tag{4}$$

$$N_1^{(d)} = \frac{1}{\lambda}\nabla;\ N_2^{(d)} = \frac{4}{\lambda}(\sigma-1)\vec{e}_2^{(1)} + \frac{1}{\lambda}\nabla(y\,\cdot);\ N_3^{(d)} = \frac{i}{\lambda}\operatorname{rot}\left(\vec{e}_3^{(1)}\cdot\right);\ N_1^{(p)} = \frac{1}{\lambda}\nabla;$$

$$N_2^{(p)} = \frac{1}{\lambda}\left[\nabla\left(\rho\frac{\partial}{\partial\rho}\right) + 4(\sigma-1)\left(\nabla - \vec{e}_3^{(2)}\frac{\partial}{\partial z}\right)\right];\ N_3^{(p)} = \frac{i}{\lambda}\operatorname{rot}\left(\vec{e}_3^{(2)}\cdot\right);\ \gamma = \sqrt{\lambda^2+\mu^2},\ -\infty<\lambda,\mu<\infty,$$

where $I_m(x)$, $K_m(x)$ is modified Bessel functions; $\vec{R}_{k,m}$, $\vec{S}_{k,m}$ are inner and outer solutions of the Lamé equation for the cylinder, respectively; $\vec{u}_k^{(-)}$, $\vec{u}_k^{(+)}$ are Lamé equation solutions for the layer σ is Poisson's ratio.

The problem solution is given in the form

$$\begin{aligned}\vec{U}_0 = &\sum_{p=1}^{2}\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} B_{k,m}^{(p)}(\lambda)\cdot\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)d\lambda +\\ &+\sum_{k=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(H_k(\lambda,\mu)\cdot\vec{u}_k^{(+)}(x,y,z;\lambda,\mu) + \widetilde{H}_k(\lambda,\mu)\cdot\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)\right)d\mu d\lambda\end{aligned} \tag{5}$$

$$\vec{U}_p = \sum_{k=1}^{3} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} A_{k,m}^{(p)}(\lambda) \cdot \vec{R}_{k,m}(\rho_p, \varphi_p, z; \lambda) d\lambda, \quad (6)$$

where $\vec{S}_{k,m}(\rho_p, \varphi_p, z; \lambda)$, $\vec{R}_{k,m}(\rho_p, \varphi_p, z; \lambda)$, $\vec{u}_k^{(+)}(x, y, z; \lambda, \mu)$ and $\vec{u}_k^{(-)}(x, y, z; \lambda, \mu)$ are basic solutions given by the formulas (4), and unknown functions $H_k(\lambda, \mu)$, $\tilde{H}_k(\lambda, \mu)$, $B_{k,m}^{(p)}(\lambda)$, $A_{k,m}^{(p)}(\lambda)$ are necessary to be found from the boundary conditions (3) and conjugation conditions (1) and (2).

The formulas [17] are used for a transition between coordinate systems:

– for a transition from solutions $\vec{S}_{k,m}$ of cylindrical coordinate system to layer solutions $\vec{u}_k^{(-)}$ (at $y>0$) and $\vec{u}_k^{(+)}$ (at $y<0$)

$$\vec{S}_{k,m}(\rho_p, \varphi_p, z; \lambda) = \frac{(-i)^m}{2} \int_{-\infty}^{\infty} \omega_{\mp}^m \cdot e^{-i\mu\bar{x}_p \pm \gamma\bar{y}_p} \cdot \vec{u}_k^{(\mp)} \cdot \frac{d\mu}{\gamma}, \quad k = 1,\ 3;$$

$$\vec{S}_{2,m}(\rho_p, \varphi_p, z; \lambda) = \frac{(-i)^m}{2} \int_{-\infty}^{\infty} \omega_{\mp}^m \cdot \left(\left(\pm m \cdot \mu - \frac{\lambda^2}{\gamma} \pm \lambda^2 \bar{y}_p \right) \vec{u}_1^{(\mp)} \mp \lambda^2 \vec{u}_2^{(\mp)} \pm \right. \quad (7)$$

$$\left. \pm 4\mu(1-\sigma) \vec{u}_3^{(\mp)} \right) \cdot \frac{e^{-i\mu\bar{x}_p \pm \gamma\bar{y}_p} d\mu}{\gamma^2},$$

where $\gamma = \sqrt{\lambda^2 + \mu^2}$, $\omega_{\mp}(\lambda, \mu) = \dfrac{\mu \mp \gamma}{\lambda}$, $m = 0, \pm 1, \pm 2, \ldots$;

– for a transition from layer solutions $\vec{u}_k^{(+)}$ and $\vec{u}_k^{(-)}$ to solutions $\vec{R}_{k,m}$ of cylindrical coordinate system

$$\vec{u}_k^{(\pm)}(x, y, z) = e^{i\mu\bar{x}_p \pm \gamma\bar{y}_p} \cdot \sum_{m=-\infty}^{\infty} (i \cdot \omega_{\mp})^m \vec{R}_{k,m}, \quad (k = 1,\ 3);$$

$$\vec{u}_2^{(\pm)}(x, y, z) = e^{i\mu\bar{x}_p \pm \gamma\bar{y}_p} \cdot \sum_{m=-\infty}^{\infty} \left[(i \cdot \omega_{\mp})^m \cdot \lambda^{-2} \left((m \cdot \mu + \bar{y}_p \cdot \lambda^2) \cdot \vec{R}_{1,m} \pm \right. \right. \quad (8)$$

$$\left. \left. \pm \gamma \cdot \vec{R}_{2,m} + 4\mu(1-\sigma) \vec{R}_{3,m} \right) \right],$$

where $\vec{R}_{k,m} = \vec{\tilde{b}}_{k,m}(\rho_p, \lambda) \cdot e^{i(m\varphi_p + \lambda z)}$; $\vec{\tilde{b}}_{1,n}(\rho, \lambda) = \vec{e}_\rho \cdot I_n'(\lambda\rho) + i \cdot I_n(\lambda\rho) \cdot \left(\vec{e}_\varphi \dfrac{n}{\lambda\rho} + \vec{e}_z \right)$;

$\vec{\tilde{b}}_{2,n}(\rho, \lambda) = \vec{e}_\rho \cdot [(4\sigma - 3) \cdot I_n'(\lambda\rho) + \lambda\rho I_n''(\lambda\rho)] + \vec{e}_\varphi i \cdot m \left(I_n'(\lambda\rho) + \dfrac{4(\sigma - 1)}{\lambda\rho} I_n(\lambda\rho) \right) + \vec{e}_z i\lambda\rho I_n'(\lambda\rho)$;

$\vec{\tilde{b}}_{3,n}(\rho, \lambda) = -\left[\vec{e}_\rho \cdot I_n(\lambda\rho) \dfrac{n}{\lambda\rho} + \vec{e}_\varphi \cdot i \cdot I_n'(\lambda\rho) \right]$; $\vec{e}_\rho$, $\vec{e}_\varphi$, $\vec{e}_z$ are unit vectors in the cylindrical coordinate system;

– for a transition from solutions of the cylinder with the number p to solutions of the cylinder with the number q

$$\vec{S}_{k,m}(\rho_p, \varphi_p, z; \lambda) = \sum_{n=-\infty}^{\infty} \vec{b}_{k,pq}^{mn}(\rho_q) \cdot e^{i(n\varphi_q + \lambda z)}, k = 1,\ 2,\ 3;$$

$$\vec{b}_{1,pq}^{mn}(\rho_q) = (-1)^n \tilde{K}_{m-n}(\lambda \ell_{pq}) \cdot e^{i(m-n)\alpha_{pq}} \cdot \vec{\tilde{b}}_{1,n}(\rho_q, \lambda);$$

$$\vec{b}_{3,pq}^{mn}(\rho_q) = (-1)^n \tilde{K}_{m-n}(\lambda \ell_{pq}) \cdot e^{i(m-n)\alpha_{pq}} \cdot \vec{\tilde{b}}_{3,n}(\rho_q, \lambda); \quad (9)$$

$$\vec{b}_{2,pq}^{mn}(\rho_q) = (-1)^n \left\{ \tilde{K}_{m-n}(\lambda \ell_{pq}) \cdot \vec{\tilde{b}}_{2,n}(\rho_q, \lambda) - \frac{\lambda}{2} \ell_{pq} \cdot \left[\tilde{K}_{m-n+1}(\lambda \ell_{pq}) + \tilde{K}_{m-n-1}(\lambda \ell_{pq}) \right] \cdot \vec{\tilde{b}}_{1,n}(\rho_q, \lambda) \right\} \cdot e^{i(m-n)\alpha_{pq}},$$

where α_{pq} is the angle between the axis x_p and the line segment.

To meet the boundary conditions at the upper layer boundary, vector (5) is equated (at $y=h$) to the given one $\vec{F}_h^0(x,z)$, introduced by the double Fourier integral. Basic solutions $\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)$ are rewritten in the Cartesian coordinate system through basic solutions $\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)$ with the help of transition formulas (7). Taking this into account, the first three equations (one for each projection) with 12 unknowns are obtained $H_k(\lambda,\mu)$, $\tilde{H}_k(\lambda,\mu)$, $B_{k,m}^{(1)}(\lambda)$, $B_{k,m}^{(2)}(\lambda)$.

To meet the boundary conditions at the lower layer boundary, vector (5) is equated (at $y=-\tilde{h}$) the given one $\vec{U}_{\tilde{h}}^0(x,z)$, introduced by the double Fourier integral. Basic solution $\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)$ is rewritten in the Cartesian coordinate system through basic solutions $\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)$ with the help of transition formulas (7). Thanks to this, three more equations are obtained.

From this system of equations, we find $H_k(\lambda,\mu)$ and $\tilde{H}_k(\lambda,\mu)$ by $B_{k,m}^{(p)}(\lambda)$.

Taking into account the conjugation conditions of the layer and inclusions, it is possible to write down three equations for each inclusion in the form of displacements (1). At the same time, using the expression $\vec{U}_0(\varphi_p,z)\big|_{\rho_p=R_p}$, it is necessary to take into account the formulas for transition from solutions $\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)$ and $\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)$ to solutions $\vec{R}_{k,m}(\rho_p,\varphi_p,z;\lambda)$ (8) and formulas for transition from solutions $\vec{S}_{k,m}(\rho_q,\varphi_q,z;\lambda)$ to solutions $\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)$ (9). Applying the stress operator to the obtained expression, it is possible to write three more equations for each inclusion in the form of stresses (2).

If there are two inclusions, then there will be 12 such equations (in displacements and stresses) with unknowns $H_k(\lambda,\mu)$, $\tilde{H}_k(\lambda,\mu)$, $B_{k,m}^{(p)}(\lambda)$, $A_{k,m}^{(p)}(\lambda)$. Excluding the $H_k(\lambda,\mu)$ and $\tilde{H}_k(\lambda,\mu)$ previously found from these equations by $B_{k,m}^{(p)}(\lambda)$ and discarding the series in m and integrals in λ, 12 infinite linear equations of the second kind to determine the unknowns $B_{k,m}^{(p)}(\lambda)$ and $A_{k,m}^{(p)}(\lambda)$ are obtained.

The determinant of this system of equations coincides with [16].

The reduction method is applied to the obtained infinite systems of equations, as a result of which the coefficients $B_{k,m}^{(p)}(\lambda)$ are found. Now $B_{k,m}^{(p)}(\lambda)$ is substituted in the expression for $H_k(\lambda,\mu)$ and $\tilde{H}_k(\lambda,\mu)$. Thus, all the unknowns of the expressions (5) and (6) will be found.

The solutions of the infinite system by the reduction method given in the paper showed its convergence, which satisfies the boundary conditions with high accuracy.

Numerical studies of the stress state

In the elastic isotropic layer (Fig. 1) there are two elastic isotropic cylindrical inclusions that are rigidly connected to it. Poisson's ratio of the layer (ABS plastic) $\sigma_0=0.38$, elastic modulus $E_0=1700$ N/mm^2, inclusions (steel) $\sigma_1=\sigma_2=0.21$, $E_1=E_2=200000$ N/mm^2. Geometric characteristics of the model: $R_1=R_2=10$ mm, $h=20$ mm, $\tilde{h}=30$ mm, $\alpha_{12}=0$. The distance between the inclusions centers is taken in two options: $L_{12}=25$ mm and $L_{12}=30$ mm.

On the upper layer boundary, stresses are given in the form of a wave $\sigma_y^{(h)}(x,z)=-10^8\cdot\left(z^2+10^2\right)^{-2}\cdot\left(x^2+10^2\right)^{-2}$, $\tau_{yx}^{(h)}=\tau_{yz}^{(h)}=0$, there are no displacements on the lower one $U_x^{(\tilde{h})}=U_y^{(\tilde{h})}=U_z^{(\tilde{h})}=0$.

The infinite system was truncated by parameter m. At $L_{12}=30$ mm, parameter $m=6$, at $L_{12}=25$ mm, parameter $m=10$.

The calculation of integrals is performed by the quadrature Filon's rule (for oscillating functions) and Simpson's rule (for functions without oscillations). The accuracy of meeting the boundary conditions at the values of m and the specified geometric parameters is 10^{-4}.

When the inclusions approach each other, the stress state in the layer almost does not change. The difference is observed only in the inclusion bodies.

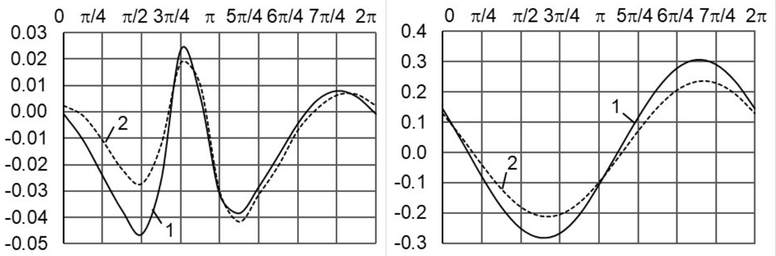
Fig. 2 shows stress graphs σ_φ and σ_z on the surface of the second inclusion at z=0 in N/mm^2.

When the second cylindrical inclusion approaches the first one, stresses σ_φ and σ_z on its surface increase. Moreover, the stresses values σ_φ increase significantly (Fig. 2, a), especially in the upper part of the inclusion, where they are maximum. In general, the increase in the inclusion stress, as it approaches the place of the stress dislocation, is natural.

Fig. 3 shows graphs of stresses σ_φ and σ_z on the surface of the first inclusion at z=0 in N/mm^2.

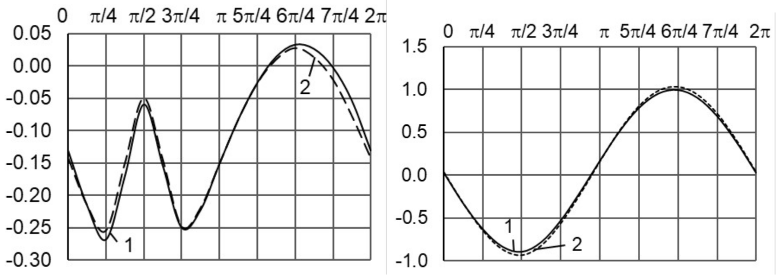
When the second cylindrical inclusion approaches the first one, the stresses σ_φ on the surface of the first cylinder increase (Fig. 3, a), and the stresses σ_z slightly decrease (Fig. 3, b), being redistributed to the second cylinder.

On the surface of the first cylinder, the maximum stresses value σ_φ occur at φ=π/4, and σ_z – at φ=π/2.



a b

Fig. 2. Stresses on the surface of the second inclusion:
a – σ_φ; b – σ_z; 1 – at L_{12}=25 mm; 2 – at L_{12}=30 mm



a b

Fig. 3. Stresses on the surface of the first inclusion:
a – σ_φ; b – σ_z; 1 – at L_{12}=25 mm; 2 – at L_{12}=30 mm

Conclusions

Based on the generalized Fourier method, a method for solving the third basic spatial problem of the elasticity theory for a layer with two longitudinal circular cylindrical inclusions is proposed. The problem is reduced to an infinite system of linear equations, which allows the reduction method to be applied to it. Numerical studies give grounds for asserting that its solution can be found with any accuracy by the given method, which is confirmed by the high accuracy of the boundary conditions.

The solution method can be used in the design of composite materials, the calculation scheme of which is a reinforced layer with specified boundary conditions in the form of stress on the upper layer boundary and displacements on the lower layer boundary.

The given comparative analysis shows that the convergence of reinforcement elements affects their stress state, in particular, the stresses σ_φ and σ_z.

The given solution method allows to obtain the stress-strain state for a layer with only two longitudinal circular cylindrical inclusions. For further development of this method, the number of inclusions can be increased to three or more. For this, it is necessary to change this algorithm by working out the connections between the two shifted coordinate systems, as well as between their basic solutions.

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Аналіз напруженого стану шару з двома циліндричними пружними включеннями й мішаними граничними умовами

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Досліджується просторова задача теорії пружності для шару з двома нескінченними круговими суцільними циліндричними включеннями, паралельними між собою й межами шару. Шар і включення є однорідними, ізотропними матеріалами, фізичні характеристики цих тіл відмінні одна від одної. Кругові циліндричні пружні включення жорстко спряжені з шаром. На верхній межі шару задана просторова функція напружень, на нижній – переміщень. Необхідно визначити напружено-деформований стан композитного тіла. При цьому розв'язання задачі базується на узагальненому методі Фур'є, де використовуються особливі формули переходу між базисними розв'язками рівняння Ламе у різних системах координат. Таким чином, шар розглядається в декартовій системі координат, включення – у локальних циліндричних. Задовольняючи граничним умовам й умовам спряження, отримано системи нескінченних інтегро-алгебраїчних рівнянь, які в подальшому зведені до лінійних алгебраїчних. Нескінченна система розв'язується методом редукції. Після знаходження невідомих можна визначити напруження в будь-якій точці пружного композиційного тіла. У чисельних дослідженнях проведено порівняльний аналіз напруженого стану на поверхнях включень за різних відстаней між ними. Аналіз показав, що при зближенні включень напружений стан у шарі практично не змінюється. Однак спостерігається суттєва його зміна в тілах включень. Так, при щільному армуванні ($(R_1 + R_2) / L > 0,5$) необхідно враховувати відстані між армуючими волокнами. При значеннях напружень від 0 до 1 і порядку системи рівнянь $m=10$ точність виконання граничних умов склала 10^{-4}. При збільшенні порядку системи точність виконання граничних умов зростатиме. Представлене аналітико-чисельне розв'язання може використовуватися для високоточного визначення напружено-деформованого стану представленого типу задач, а також як еталонне для задач, що базуються на чисельних методах.

***Ключові слова**: композит, циліндричні включення в шарі, узагальнений метод Фур'є.*

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