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BIFURCATIONS AND STABILITY OF NONLINEAR VIBRATIONS OF A THREE-LAYER COMPOSITE SHELL WITH MODERATE AMPLITUDES

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The authors derived a mathematical model of geometrically nonlinear vibrations of sandwich shells, which describes the vibrations of the structure with amplitudes comparable to its thickness. The high-order shear theory is used for the derivation of this model. Rotational inertia is also taken into account. The middle layer of the structure is a honeycomb core made using additive FDM technologies. In addition, each shell layer is described by five variables (three displacement projections and two rotation angles of the normal to the middle surface). The total number of unknown variables is fifteen. To obtain a model of nonlinear vibrations of the structure, the method of given forms is used. The potential energy, which takes into account the quadratic, cubic, and fourth powers of the generalized displacements of the structure, is derived. All generalized displacements are expanded into series by generalized coordinates and eigenforms, which are recognized as basis functions. It is shown that the mathematical model of shell vibrations is a system of nonlinear non-autonomous ordinary differential equations. Nonlinear periodic vibrations and their bifurcations have been studied using a numerical procedure, which is a combination of the continuation method and the shooting technique. The shooting technique takes into account periodicity conditions expressed by a system of nonlinear algebraic equations with respect to the initial conditions of periodic vibrations. These equations are solved using Newton's method. The properties of nonlinear periodic vibrations and their bifurcations in the area of subharmonic resonances are numerically studied. Stable subharmonic vibrations of the second order, which undergo a saddle-node bifurcation, are discovered. An infinite sequence of bifurcations leading to chaotic vibrations is not detected.

***Keywords**: double curved shell, additive technologies, honeycomb structure, bifurcation behavior.*

Introduction

Multilayer structures have proven themselves well in aerospace engineering field, where they have been used for over 70 years. This is primarily because these designs have high flexural stiffness at low weight. In addition, their parts, such as honeycomb structures, are manufactured using additive technologies [1, 2], which has a number of advantages. For example, these technologies make it possible to produce parts with internal cavities, which lightweights the construction significantly. Considering this, additive technologies are also used for the manufacture of aircraft parts [3, 4].

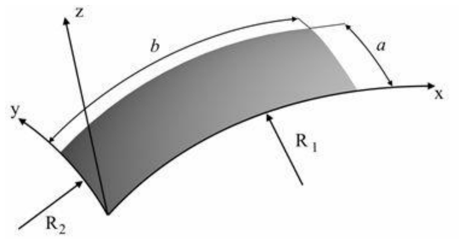
A lot of effort has been devoted to the study of multilayer structures with honeycomb structure. The partial derivative equations of the free vibrations of the conical shell are derived by using variational methods [5]. With the help of Mindlin theory, a composite three-layer structure is numerically studied [6]. The high-order shear theory by Frostig et al. [7] is used for the analysis of multilayer beams with elastic structure, and one by Malekzadeh et al. – for the analysis of vibrations of multilayer plates with viscoelastic structure [8]. Free linear vibrations of multilayer panels with flexible structure are considered in [9]. Transient processes in a multilayer panel are analyzed by using an efficient interlayer formulation of the problem [10].

In this paper, a study on sandwich structures is presented. Main novelty of the paper is contained in the results of numerical modeling of bifurcations of nonlinear forced vibrations in the area of subharmonic resonances of double curved sandwich shells with honeycomb core, manufactured using additive FDM technologies. To achieve this goal, a new mathematical model of geometrically nonlinear dynamic deformation of double curved sandwich shells, which uses high-order shear theory, has been built. The stress-strain state of each shell layer is described by five parameters (three displacement projections and two rotation angles of the normal to the middle surface). Using the method of given forms, the specified mathematical model is reduced to a system of nonlinear ordinary differential equations. Bifurcations of periodic vibrations of the obtained dynamic system are studied by a numerical method, which is a combination of the continuation method and the shooting technique.

Problem statement and basic equations

The object of study is a double curved sandwich shell (Fig. 1). Its upper and lower faces are made of carbon fiber, and the middle one is a honeycomb core manufactured using FDM technologies from ULTEM 9085 material. The upper, middle and lower layers of the shell have constant thicknesses h_t, h_c, h_b. Deformation of the structure is considered in curvilinear coordinates (x, y). The radii of curvature of these coordinate lines are denoted by R_1 and R_2. Z axis is directed perpendicular to the axes x, y (Fig. 1). We use three transverse coordinates z_t, z_c, z_b, associated with the three middle surfaces of the layers. The lengths of the two curved sides of the shell are a and b (Fig. 1). The main geometric parameters of honeycomb cell (Fig. 1) are l_1, l_2, h_c, ψ, where h_c is the thickness of the honeycomb structure.

The shell performs forced vibrations under periodic excitation by a concentrated force $F_0 \cos(\Omega t)$. Its magnitude is large enough to result in the amplitudes of shell vibrations comparable to its thickness. Therefore, the shell undergoes geometrically nonlinear deformation, which will be discussed below.



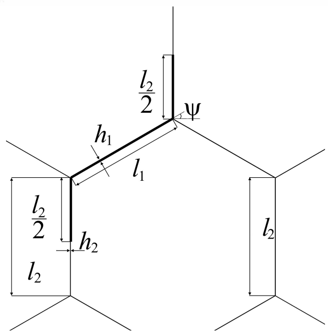


Fig. 1. Design of a sandwich shell and a single cell of a honeycomb structure

The stress-strain state of the structure is described by the projections of displacements of the points of each layer onto the coordinate axes (x, y, z): $u_1^{(t)}, u_2^{(t)}, u_3^{(t)}, u_1^{(c)}, u_2^{(c)}, u_3^{(c)}, u_1^{(b)}, u_2^{(b)}, u_3^{(b)}$.

The upper and lower layers of the shell are orthotropic. They satisfy Hooke's law in the following form:

$$\begin{bmatrix} \sigma_{xx}^{(j)} \\ \sigma_{yy}^{(j)} \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} \\ \overline{C}_{12} & \overline{C}_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(j)} \\ \varepsilon_{yy}^{(j)} \end{bmatrix};$$

$$\sigma_{xy}^{(j)} = 2\overline{C}_{66}\varepsilon_{xy}^{(j)}; \quad \sigma_{xz}^{(j)} = 2\overline{C}_{55}\varepsilon_{xz}^{(j)}; \quad \sigma_{yz}^{(j)} = 2\overline{C}_{44}\varepsilon_{yz}^{(j)}; \quad j = b, t, \tag{1}$$

where $\sigma_{xx}^{(j)}, \sigma_{yy}^{(j)}, \sigma_{xy}^{(j)}, \sigma_{xz}^{(j)}, \sigma_{yz}^{(j)}, \varepsilon_{xx}^{(j)}, \varepsilon_{yy}^{(j)}, \varepsilon_{xy}^{(j)}, \varepsilon_{xz}^{(j)}, \varepsilon_{yz}^{(j)}$ are elements of stress and strain tensors.

The honeycomb structure is presented as an equivalent orthotropic medium by homogenization of the structure [11]. This paper does not consider the calculation of the parameters of the homogenized medium. As a result of such modeling, the matrix of Hooke's law is calculated

$$\begin{bmatrix} \sigma_{xx}^{(c)} \\ \sigma_{yy}^{(c)} \\ \sigma_{zz}^{(c)} \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} \\ \overline{C}_{21} & \overline{C}_{22} & \overline{C}_{23} \\ \overline{C}_{31} & \overline{C}_{32} & \overline{C}_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{(c)} \\ \varepsilon_{yy}^{(c)} \\ \varepsilon_{zz}^{(c)} \end{bmatrix};$$

$$\sigma_{yz}^{(c)} = 2C_{44}\varepsilon_{yz}^{(c)}; \quad \sigma_{xz}^{(c)} = 2C_{55}\varepsilon_{xz}^{(c)}; \quad \sigma_{xy}^{(c)} = 2C_{66}\varepsilon_{xy}^{(c)}, \tag{2}$$

where $\sigma_{xx}^{(c)}, \sigma_{yy}^{(c)}, \sigma_{zz}^{(c)}, \sigma_{yz}^{(c)}, \sigma_{xz}^{(c)}, \sigma_{xy}^{(c)}, \varepsilon_{xx}^{(c)}, \varepsilon_{yy}^{(c)}, \varepsilon_{zz}^{(c)}, \varepsilon_{yz}^{(c)}, \varepsilon_{xz}^{(c)}, \varepsilon_{xy}^{(c)}$ are elements of the stresses and strains tensor of honeycomb structure.

To describe the state of the deforming shell, the high-order shear theory [12] is used. The displacements of the points of the upper and lower layers of the shell in the projections on the axes (x, y, z) are given as follows:

$$u_1^{(i)} = u^{(i)}\left(1+\frac{z_i}{R_1}\right) + z_i\phi_1^{(i)} + z_i^2\varphi_1^{(i)}; \quad u_2^{(i)} = v^{(i)}\left(1+\frac{z_i}{R_2}\right) + z_i\phi_2^{(i)} + z_i^2\varphi_2^{(i)}; \quad u_3^{(i)} = w^{(i)}; \quad i=t,\, b, \tag{3}$$

where $u^{(i)}$, $v^{(i)}$, $w^{(i)}$ are displacements projections of the points of the middle surfaces of the layers onto the coordinate axes (x, y, z_i); $i=t, b$; $\phi_1^{(i)}$, $\phi_2^{(i)}$ are rotation angles of the normal to the middle surface of the corresponding layer; $\varphi_1^{(i)}$, $\varphi_2^{(i)}$ are unknown functions to be determined.

The displacements field of the homogenized core is given as follows:

$$u_1^{(c)} = u^{(c)}\left(1+\frac{z_c}{R_1}\right) + z_c\phi_1^{(c)} + z_c^2\varphi_1^{(c)} + z_c^3\gamma_1^{(c)}; \quad u_2^{(c)} = v^{(c)}\left(1+\frac{z_c}{R_2}\right) + z_c\phi_2^{(c)} + z_c^2\varphi_2^{(c)} + z_c^3\gamma_2^{(c)};$$

$$u_3^{(c)} = w^{(c)} + z_c w_1^{(c)} + z_c^2 w_2^{(c)}, \tag{4}$$

where $u^{(c)}$, $v^{(c)}$, $w^{(c)}$ are displacements projections of the points of the middle surface of the homogenized core; $\phi_1^{(c)}$, $\phi_2^{(c)}$ are rotation angles of the normal to the middle surface; $\varphi_1^{(c)}$, $\varphi_2^{(c)}$, $\gamma_1^{(c)}$, $\gamma_2^{(c)}$ are unknown functions to be determined.

The following boundary conditions and conditions of displacements continuity are fulfilled on the upper and lower sides of the shell and on the surfaces between the layers:

$$\varepsilon_{xz}^{(t)}\Big|_{z_t=0.5h_t} = \varepsilon_{yz}^{(t)}\Big|_{z_t=0.5h_t} = \varepsilon_{xz}^{(b)}\Big|_{z_t=-0.5h_b} = \varepsilon_{yz}^{(b)}\Big|_{z_t=-0.5h_b} = 0;$$

$$u_i^{(t)}(z_t = -0{,}5h_t) = u_i^{(c)}(z_c = 0{,}5h_c); \; u_i^{(b)}(z_b = 0{,}5h_b) = u_i^{(c)}(z_c = -0{,}5h_c); \quad i=1, 2, 3. \tag{5}$$

Using conditions (5), the unknown parameters of the expansions are found (3, 4): $\varphi_1^{(t)}$, $\varphi_2^{(t)}$, $\varphi_1^{(c)}$, $\varphi_2^{(c)}$, $\gamma_1^{(c)}$, $w_1^{(c)}$, $w_2^{(c)}$.

The shell is considered to be fixed along the line ∂D. Then the boundary conditions take the following form:

$$u^{(i)}\Big|_{\partial D} = v^{(i)}\Big|_{\partial D} = w^{(i)}\Big|_{\partial D} = \phi_1^{(i)}\Big|_{\partial D} = \phi_2^{(i)}\Big|_{\partial D} = 0; \quad i=t,\, c,\, b. \tag{6}$$

Geometrically nonlinear deformation is described by a nonlinear relation between deformations and displacements. The general case of such relations in curvilinear coordinates is given in the monograph [13]. These relations are used further. Substitution of displacements (3, 4) into these nonlinear relations allows to obtain the following expansions for the elements of strain tensors:

$$\varepsilon_{xx}^{(i)} = \varepsilon_{1,0}^{(i)} + z_i k_{1,0}^{(i)} + z_i^2 k_{1,1}^{(i)} + z_i^3 k_{1,2}^{(i)}; \quad \varepsilon_{yy}^{(i)} = \varepsilon_{2,0}^{(i)} + z_i k_{2,0}^{(i)} + z_i^2 k_{2,1}^{(i)} + z_i^3 k_{2,2}^{(i)};$$

$$\varepsilon_{xy}^{(i)} = \varepsilon_{12,0}^{(i)} + z_i k_{12,0}^{(i)} + z_i^2 k_{12,1}^{(i)} + z_i^3 k_{12,2}^{(i)}; \quad \varepsilon_{xz}^{(i)} = \varepsilon_{13,0}^{(i)} + z_i k_{13,0}^{(i)} + z_i^2 k_{13,1}^{(i)} + z_i^3 k_{13,2}^{(i)};$$

$$\varepsilon_{yz}^{(i)} = \varepsilon_{23,0}^{(i)} + z_i k_{23,0}^{(i)} + z_i^2 k_{23,1}^{(i)} + z_i^3 k_{23,2}^{(i)}; \quad i=t,\, c,\, b;$$

$$\varepsilon_{zz}^{(c)} = \varepsilon_{3,0}^{(c)} + z_c k_{3,0}^{(c)}. \tag{7}$$

All nonlinear terms in relations (7) describing geometrically nonlinear deformation are concentrated in the terms at the zero power of z_i. These terms describe the deformation of the middle surfaces of the shell layers. All other terms at non-zero terms of z_i depend linearly on the displacement projections.

The potential energies of deformation of the faces of the shell have the following form:

$$U_i = 0.5\int_{V_i}\left(\sigma_{xx}^{(i)}\varepsilon_{xx}^{(i)} + \sigma_{yy}^{(i)}\varepsilon_{yy}^{(i)} + \sigma_{xy}^{(i)}\varepsilon_{xy}^{(i)} + \sigma_{xz}^{(i)}\varepsilon_{xz}^{(i)} + \sigma_{yz}^{(i)}\varepsilon_{yz}^{(i)}\right)\left(1+\frac{z_i}{R_1}\right)\left(1+\frac{z_i}{R_2}\right)R_1R_2\,d\psi\,d\theta\,dz_i =$$

$$= \int_{V_i}\left(\overline{C}_{11}\varepsilon_{xx}^{(i)2} + \overline{C}_{22}\varepsilon_{yy}^{(i)2} + 2\overline{C}_{12}\varepsilon_{yy}^{(i)}\varepsilon_{xx}^{(i)} + 2\overline{C}_{66}\varepsilon_{xy}^{(i)2} + 2\overline{C}_{55}\varepsilon_{xz}^{(i)2} + 2\overline{C}_{44}\varepsilon_{yz}^{(i)2}\right)\left(1+\frac{z_i}{R_1}\right)\left(1+\frac{z_i}{R_2}\right)R_1R_2\,d\psi\,d\theta\,dz_i; \; i=t,\, b, \tag{8}$$

where $\psi=x/R_1$; $\theta=y/R_2$; V_i is the volume occupied by the shell layer.

Now let's consider the potential energy of the homogenized layer that corresponds to Hooke's law (2). The potential energy is given as follows:

$$U_c = 0.5\int_{V_c}\left(\sigma_{xx}^{(c)}\varepsilon_{xx}^{(c)} + \sigma_{yy}^{(c)}\varepsilon_{yy}^{(c)} + \sigma_{zz}^{(c)}\varepsilon_{zz}^{(c)} + \sigma_{xy}^{(c)}\varepsilon_{xy}^{(c)} + \sigma_{xz}^{(c)}\varepsilon_{xz}^{(c)} + \sigma_{yz}^{(c)}\varepsilon_{yz}^{(c)}\right)\left(1+\frac{z_c}{R_1}\right)\left(1+\frac{z_c}{R_2}\right)R_1R_2\, d\psi\, d\theta\, dz_c =$$

$$= 0.5\int_{V_c}\left(C_{11}\varepsilon_{xx}^{(c)2} + C_{22}\varepsilon_{yy}^{(c)2} + 2C_{12}\varepsilon_{xx}^{(c)}\varepsilon_{yy}^{(c)} + 2C_{55}\varepsilon_{xyz}^{(c)2} + 2C_{66}\varepsilon_{xy}^{(c)2} + 2C_{44}\varepsilon_{yz}^{(c)2}\right)\left(1+\frac{z_c}{R_1}\right)\left(1+\frac{z_c}{R_2}\right)R_1R_2\, d\psi\, d\theta\, dz_c +$$

$$+ 0.5\int_{V_c}\left(C_{33}\varepsilon_{zz}^{(c)2} + 2C_{23}\varepsilon_{yy}^{(c)}\varepsilon_{zz}^{(c)} + 2C_{13}\varepsilon_{xx}^{(c)}\varepsilon_{zz}^{(c)}\right)\left(1+\frac{z_c}{R_1}\right)\left(1+\frac{z_c}{R_2}\right)R_1R_2\, d\psi\, d\theta\, dz_c. \qquad (9)$$

The kinetic energies of the layers of the T_i structure are given as follows:

$$T_i = 0.5\int_{V_i}\rho_i\left(\dot{u}_1^{(i)2} + \dot{u}_2^{(i)2} + \dot{u}_3^{(i)2}\right)\left(1+\frac{z_i}{R_1}\right)\left(1+\frac{z_i}{R_2}\right)R_1R_2\, d\psi\, d\theta\, dz_i,\ i=t,\ c,\ b, \qquad (10)$$

where ρ_i is the material density of the i-th layer; $\dot{u}_1^{(i)} = \dfrac{\partial u_1^{(i)}}{\partial t}$.

The equation of forced nonlinear vibrations

The forced nonlinear vibrations of the shell are expanded into series by the forms of linear vibrations. Therefore, before analyzing a nonlinear system, it is extremely important to study linear vibrations. For this, the Rayleigh-Ritz method [14] is used. The linear vibrations of the shell are given as follows:

$$\begin{bmatrix} u^{(i)} \\ v^{(i)} \\ w^{(i)} \\ \phi_1^{(i)} \\ \phi_2^{(i)} \end{bmatrix} = \begin{bmatrix} U_i(x,y) \\ V_i(x,y) \\ W_i(x,y) \\ X_i(x,y) \\ Y_i(x,y) \end{bmatrix}\cos(\omega t);\ i=b,\ c,\ t, \qquad (11)$$

where ω is the frequency of linear vibrations; $U_i(x, y)$, $V_i(x, y)$, $W_i(x, y)$, $X_i(x, y)$, $Y_i(x, y)$ are functions to be calculated that satisfy the boundary conditions (6). These functions will be given in the form of the following expansions:

$$U_i = \sum_{\mu=1}^{N_i^{(u)}}\sum_{j=1}^{L_i^{(u)}} A_{i\mu j}^{(u)} U_x^{(\mu)}(x) U_y^{(j)}(y);\quad V_i = \sum_{\mu=1}^{N_i^{(v)}}\sum_{j=1}^{L_i^{(v)}} A_{i\mu j}^{(v)} V_x^{(\mu)}(x) V_y^{(j)}(y);\quad W_i = \sum_{\mu=1}^{N_i^{(w)}}\sum_{j=1}^{L_i^{(w)}} A_{i\mu j}^{(w)} W_x^{(\mu)}(x) W_y^{(j)}(y);$$

$$X_i = \sum_{\mu=1}^{N_i^{(f)}}\sum_{j=1}^{L_i^{(f)}} A_{i\mu j}^{(f)} F_x^{(\mu)}(x) F_y^{(j)}(y);\quad Y_i = \sum_{\mu=1}^{N_i^{(g)}}\sum_{j=1}^{L_i^{(g)}} A_{i\mu j}^{(g)} G_x^{(\mu)}(x) G_y^{(j)}(y), \qquad (12)$$

where $A = (A_{t11}, A_{t12}, ..., A_{bN_b^{(g)}L_b^{(g)}})$ is the vector of unknown parameters of linear vibrations, which will be given as follows: $A = (A_1, A_2, ..., A_{N_*})$; $U_x^{(1)}(x)$, $U_x^{(2)}(x)$, … are basic functions.

The potential energy of the entire structure is written as follows:

$$U_\Sigma = U_t + U_c + U_b. \qquad (13)$$

Since linear vibrations are studied, nonlinear terms with respect to displacement projections in expansion (7) are discarded when calculating the potential energy. The kinetic energy of the composite structure is given as follows:

$$T_\Sigma = T_t + T_c + T_b. \qquad (14)$$

The Rayleigh-Ritz method is used to analyze linear vibrations. It allows to reduce the analysis to the generalized problem of eigenvalues, from which the eigenfrequencies and forms of vibrations are determined.

Forced vibrations of the structure are excited by a periodic concentrated force applied at a point with coordinates $x=x_0$; $y=y_0$. This force is given as follows:

$$F = F_0 \cos(\Omega t)\delta(x - x_0, y - y_0), \tag{15}$$

where F_0, Ω are amplitude and frequency of the excitation force, respectively; $\delta(x–x_0, y–y_0)$ is the delta function.

Nonlinear vibrations are expanded into series by the forms of natural vibrations. These expansions are given as follows:

$$w^{(i)} = \sum_{j=1}^{N_w} q_{N_w(i-1)+j} W_{i,j}(x,y);\ \phi_1^{(i)} = \sum_{j=1}^{N_{\phi 1}} q_{3N_w + N_{\phi 1}(i-1)+j} X_{i,j}(x,y);\quad \phi_2^{(i)} = \sum_{j=1}^{N_{\phi 2}} q_{3N_w+3N_{\phi_1}+N_{\phi_2}(i-1)+j} Y_{i,j}(x,y);$$

$$u^{(i)} = \sum_{j=1}^{N_u} q_{3N_w+3N_{\phi_1}+3N_{\phi_2}+N_u(i-1)+j} U_{i,j}(x,y);\quad v^{(i)} = \sum_{j=1}^{N_v} q_{3N_w+3N_{\phi_1}+3N_{\phi_2}+3N_u+N_v(i-1)+j} V_{i,j}(x,y);\ i=1,\ \dots,\ 3, \tag{16}$$

where $W_{i,j}$, $X_{i,j}$, $Y_{i,j}$, $U_{i,j}$, $V_{i,j}$ are vibrations eigenforms; i is the layer number; j is the number of eigenform; $q=(q_1, \dots, q_{N*})$ is the vector of generalized coordinates.

To obtain the generalized forces, the work is written as: $\delta A = -\int_D F\delta w^{(1)} dxdy$. The generalized force O_j, which corresponds to the generalized coordinate q_j; $j=1, \dots, N_w$, takes the following form: $Q_j = H_j \cos(\Omega t)$, where $H_j = -F_0 W_{1,j}(x_0, y_0)$.

Expansions (16) are substituted into kinetic and potential energies (8–10). Then the kinetic energy is given in a quadratic form with respect to the generalized velocities $T_\Sigma = T_\Sigma(\dot q_1,...,\dot q_{N_*})$. Potential energy can be given as follows:

$$U_\Sigma = U_\Sigma^{(2)}(q_1,...,q_{N_*}) + U_\Sigma^{(3)}(q_1,...,q_{N_*}) + U_\Sigma^{(4)}(q_1,...,q_{N_*}), \tag{17}$$

where $U_\Sigma^{(2)}(q_1,...,q_{N_*})$ are quadratic terms to the vector of generalized coordinates; $U_\Sigma^{(3)}(q_1,...,q_{N_*})$ are cubic terms to the vector of generalized coordinates; $U_\Sigma^{(4)}(q_1,...,q_{N_*})$ are terms of the fourth power to the vector of generalized coordinates.

The kinetic and potential energies of the structure are substituted into the Lagrange equations. Then the equations of the structure motion take on the following matrix form:

$$\begin{bmatrix} \mathrm{M}_{11} & \mathrm{M}_{12} \\ \mathrm{M}_{21} & \mathrm{M}_{22} \end{bmatrix} \begin{bmatrix} \ddot q_1 \\ \ddot q_2 \end{bmatrix} + \begin{bmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & \mathrm{K}_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \Re_1^{(2)}(q_1,q_2) + \Re_1^{(3)}(q_1,q_2) \\ \Re_2^{(2)}(q_1,q_2) + \Re_2^{(3)}(q_1,q_2) \end{bmatrix} = \begin{bmatrix} \mathrm{H} \\ 0 \end{bmatrix} \cos(\Omega t), \tag{18}$$

where $H=[H_1, \dots, H_{Nw}, 0, 0, \dots]$; $\Re_1^{(2)}(q_1, q_2)$, $\Re_2^{(2)}(q_1, q_2)$ is the vector functions of quadratic polynomials with respect to generalized coordinates; $\Re_1^{(3)}(q_1, q_2)$, $\Re_2^{(3)}(q_1, q_2)$ is the vector functions of cubic polynomials.

As the faces of the sandwich shell are very thin, and the density of the homogenized honeycomb core is extremely low, all matrix elements M_{12}, M_{22}, M_{21} are close to zero. In further analysis, these terms are assumed zero. Then the dynamic system (18) is given as follows:

$$\mathrm{M}_{11}\ddot q_1 + \mathrm{K}_{11} q_1 + \mathrm{K}_{12} q_2 + \Re_1^{(2)}(q_1,q_2) + \Re_1^{(3)}(q_1,q_2) = \mathrm{H}\cos(\Omega t);$$

$$\mathrm{K}_{21} q_1 + \mathrm{K}_{22} q_2 + \Re_2^{(2)}(q_1,q_2) + \Re_2^{(3)}(q_1,q_2) = 0. \tag{19}$$

The second matrix equation is considered. At the first stage, we discard the nonlinear terms and write its solution as follows:

$$q_2 = Rq_1;\ R = -\mathrm{K}_{22}^{-1}\mathrm{K}_{21}.$$

The solution (19) is substituted into the second equation (18). The obtained ratios are given as follows:

$$q_2 = Rq_1 - \mathrm{K}_{22}^{-1}\left[\Re_2^{(2)}(q_1, Rq_1) + \Re_2^{(3)}(q_1, Rq_1)\right]. \tag{20}$$

Equation (20) is substituted into the first matrix equation (18). As a result, we will get the following system of differential equations:

$$\sum_{j=1}^{N_1} m_{ij}\ddot{q}_j + \beta_i\dot{q}_i + \sum_{j=1}^{N_1} K_{ij}q_j + \sum_{\nu=1}^{N_1}\sum_{j=1}^{\nu}\alpha_{\nu j}^{(i)}q_\nu q_j + \sum_{\nu=1}^{N_1}\sum_{j=1}^{\nu}\sum_{j_1=1}^{j}\beta_{\nu j j_1}^{(i)} q_\nu q_j q_{j_1} = H_i\cos(\Omega t), \qquad (21)$$

where $N_1=\dim(q_1)$; m_{ij} are elements of the mass matrix; $K_{11}^{(\Sigma)}=\{K_{ij}\}$ is the stiffness matrix, the elements of which are derived from the following matrix equation: $K_{11}^{(\Sigma)} = K_{11} - K_{12}K_{22}^{-1}K_{21}$; $\alpha_{\nu j}^{(i)}$, $\beta_{\nu j j1}^{(i)}$ are coefficients of nonlinear terms of the dynamic system.

The dynamic system (22) is given with respect to dimensionless variables and parameters

$$\vartheta_j = \frac{q_j}{h_c}; \quad \tau = \omega_1 t; \quad \bar\Omega = \frac{\Omega}{\omega_1}; \quad \tau_1 = \bar\Omega\tau; \qquad (22)$$

$$\bar m_{ij} = \frac{m_{ij}}{m_0}; \quad \bar\beta_i = \frac{\beta_i}{m_0\omega_1^2}; \quad \bar K_{ij} = \frac{K_{ij}}{m_0\omega_1^2}; \quad \bar\alpha_{\nu j}^{(i)} = \frac{\alpha_{\nu j}^{(i)}h_c}{m_0\omega_1^2}; \quad \bar\beta_{\nu j j_1}^{(i)} = \frac{\beta_{\nu j j_1}^{(i)}h_c^2}{m_0\omega_1^2};$$

$$\bar H_i = \frac{H_i}{m_0\omega_1^2 h_c} = \frac{F_0 W_{1,j}(x_0, y_0)}{m_0\omega_1^2 h_c} = \bar F W_{1,j}(x_0, y_0); \quad m_0 = \rho_c h_c ab,$$

where ω_1 is the first eigenfrequency of vibrations of the structure.

The dynamic system (22) with respect to dimensionless variables and parameters takes the following form:

$$\bar\Omega^2\sum_{j=1}^{N_1}\bar m_{ij}\vartheta''_j + \bar\Omega\bar\beta_i\vartheta'_j\sum_{j=1}^{N_1}\bar K_{ij}\vartheta_j + \sum_{\nu=1}^{N_1}\sum_{j=1}^{\nu}\bar\alpha_{\nu j}^{(i)}\vartheta_\nu\vartheta_j + \sum_{\nu=1}^{N_1}\sum_{j=1}^{\nu}\sum_{j_1=1}^{j}\bar\beta_{\nu j j_1}^{(i)}\vartheta_\nu\vartheta_j\vartheta_{j_1} = \bar H_i\cos(\tau_1), \qquad (23)$$

where $\bar{\mathrm{M}} = \{\bar m_{ij}\}$.

Numerical analysis of nonlinear vibrations

The numerical modeling of nonlinear vibrations of a sandwich hyperbolic paraboloid shell (Fig. 2) is considered.

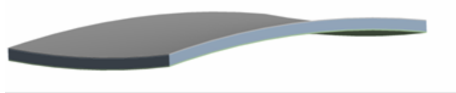


Fig. 2. Sandwich hyperbolic paraboloid shell

The geometric parameters of the honeycomb core are as follows:

$$l_1=6.1054 \text{ mm}; \; l_2=3.0527 \text{ mm}; \; \theta=60^\circ; \; l_c=10 \text{ mm}; \; \bar h_c=0.4 \text{ mm}, \qquad (24)$$

where $\bar h_c$ is the thickness of the honeycomb walls; l_c is the height of the honeycomb structure.

The honeycomb core is replaced by a homogenized layer with the following mechanical characteristics:

$$E_{11}=2.91 \text{ MPa}; \; E_{22}=2.91 \text{ MPa}; \; E_{33}=215.1 \text{ MPa}; \; \nu_{12}=0.972; \; \nu_{23}=0.0051; \; \nu_{13}=0.0042;$$

$$G_{12}=1.118 \text{ MPa}; \; G_{23}=39.1 \text{ MPa}; \; G_{13}=39.1 \text{ MPa}; \; \rho_c=253.189 \text{ kg/m}^3. \qquad (25)$$

The upper and lower layers are made of carbon fiber, which satisfies Hooke's law. The engineering constants of this material are as follows:

$$E_x=160\times10^9 \text{ Pa}; \; E_y=6.8\times10^9 \text{ Pa}; \; \nu_{xy}=0.32; \; \nu_{yx}=0.0136;$$

$$G_{xy}=800\times10^9 \text{ Pa}; \; G_{xz}=G_{yz}=4\times10^9 \text{ Pa}; \; \rho_t=\rho_b=1400 \text{ kg/m}^3. \qquad (26)$$

The geometric parameters of the design are as follows:

$$a=0.22 \text{ m}; \; b=0.33 \text{ m}; \; R_1=-R_2=0.6 \text{ m}; \; h_t=h_b=10^{-3} \text{ m}; \; h_c=10^{-2} \text{ m}.$$

Periodic vibrations and their bifurcations were studied using the shooting technique [15–18]. It allows to reduce the problem to a system of nonlinear algebraic equations regarding the initial conditions of periodic vibrations, which is solved by Newton's method. To calculate the Jacobi matrices of the Newton method, special systems of differential equations are derived. Their solutions are elements of the Jacobi matrices [19, 20]. For calculations of frequency responses, the shooting method is used alongside with the method of continuation of the solution according to the parameter [15–18].

To assess the stability and bifurcations of periodic vibrations, multipliers were calculated [15–18].

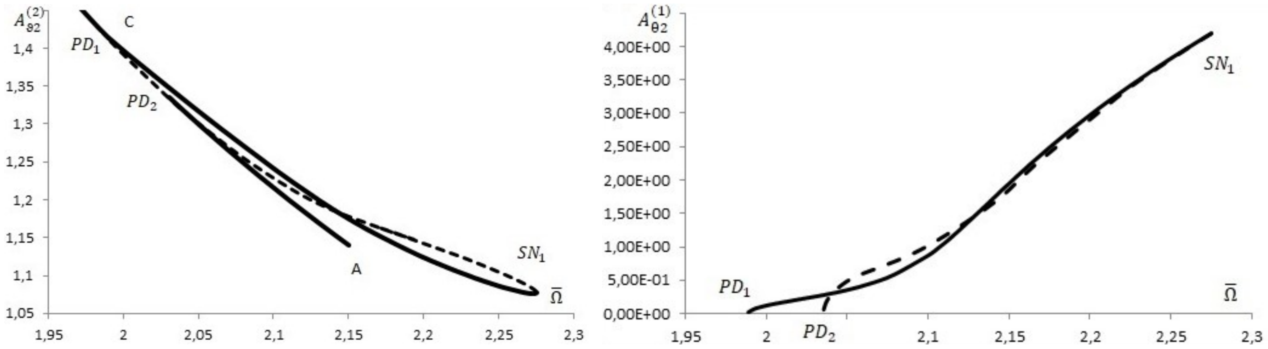
We will numerically study periodic vibrations in the region of subharmonic resonances that satisfy the condition:

$$\overline{\Omega} = 2\overline{\omega}_1 + \varepsilon\sigma ,$$

where $\overline{\omega}_1$ –is the first dimensionless frequency of vibrations of the structure; $0<\varepsilon<<1$ is the small parameter; σ is the expansion

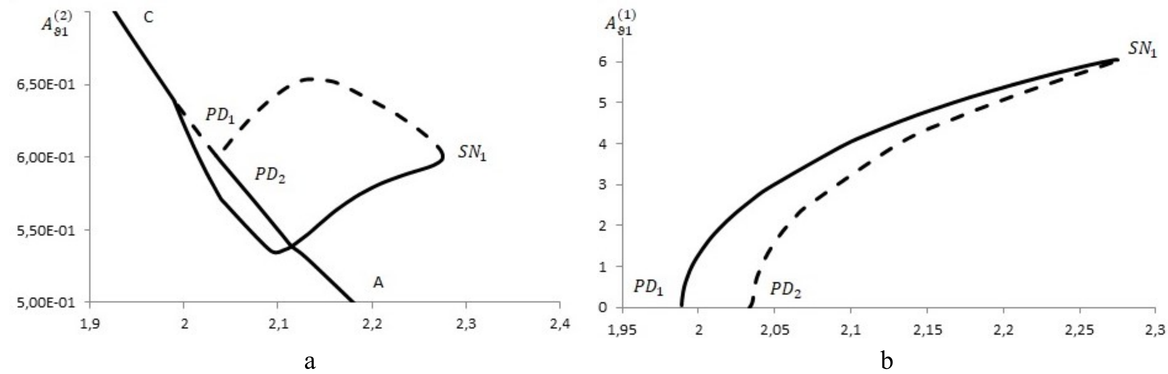
The results of calculations of nonlinear vibrations are given on frequency responses (Figs. 3–4). Fig. 3, a shows the dependence of the second harmonic of the Fourier series of subharmonic vibrations $\vartheta_2(\tau_1)$ on the dimensionless frequency of the excitation $\overline{\Omega}$. Fig. 4, a shows the same dependence for the second harmonic of the Fourier series of periodic vibrations $\vartheta_1(\tau_1)$. The *AC* curve shown in Fig. 3, a describes harmonic periodic vibrations in the region of the main resonance, undergoing two period-doubling bifurcations PD_1 and PD_2. Due to these bifurcations, subharmonic solutions of the second order, which are described by a solid line located between the bifurcation points PD_1 and SN_1 appear. Such persistent subharmonic vibrations undergo saddle-node bifurcation SN_1, which results in these fluctuations becoming unstable. Such unstable subharmonic vibrations are described by a dashed line between the points SN_1 and PD_2. In addition, they merge into harmonic ones at the bifurcation point of period-doubling PD_2, in which unstable harmonic vibrations are transformed into stable ones. The just described dynamic behavior of vibrations can be traced in Fig. 3, a, where harmonic $A_{\vartheta 2}^{(2)}$ is shown.

Subharmonic vibrations born at bifurcation points of period doubling are polyharmonic. The first harmonics of the Fourier series of vibrations $\vartheta_1(\tau_1)$ and $\vartheta_2(\tau_1)$ are shown in Fig. 3, b and Fig. 4, b. Such harmonics occur only in subharmonic vibrations. As can be seen from Fig. 3–4, the first harmonic has much larger amplitudes compared to the second one. As an example, Fig. 5 shows subharmonic vibrations of the second order $\dot{\vartheta}_2(\tau_1)$.



a b

Fig. 3. Frequency responses of the harmonics of the Fourier series of vibrations $\vartheta_2(\tau_1)$:
a – the second harmonic of the Fourier series is shown on the y-axis $A_{\vartheta 2}^{(2)}$;
b – the y-axis shows the first harmonic of the Fourier series $A_{\vartheta 2}^{(1)}$



a b

Fig. 4. Frequency responses of the harmonics of the Fourier series of vibrations $\vartheta_1(\tau_1)$:
a – the second harmonic of the Fourier series is shown on the y-axis $A_{\vartheta 1}^{(2)}$;
b – the y-axis shows the first harmonic of the Fourier series $A_{\vartheta 1}^{(1)}$

Conclusions

In the course of the research, a new mathematical model of nonlinear vibrations of double curved sandwich shells was derived. Honeycomb core is manufactured using FDM additive technologies. It is proved that the deformed state of the shell is described by a high-order shear theory, and the dynamic behavior of each layer of the shell is described by five values (three displacement projections and three rotation angles of the normal to the middle surface).

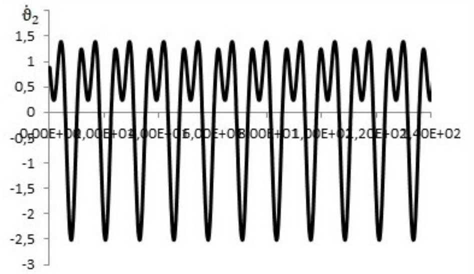


Fig. 5. Subharmonic vibrations of the second order at the frequency of the excitation action $\overline{\Omega}$ ***=2.187***

Using the method of given forms, a system of nonlinear ordinary differential equations that describes the nonlinear vibrations of a sandwich structure is derived. As the upper and lower layers are thin, and the density of the honeycomb core is small, some of the elements of the mass matrix have elements close to zero. As a result, part of the generalized coordinates behaves quasi-statically. These generalized coordinates are expressed in terms of other generalized coordinates that describe high-frequency vibrations. This approach makes it possible to reduce the dimensionality of the dynamic system describing nonlinear vibrations.

To investigate periodic vibrations, the resulting system of nonlinear ordinary differential equations is numerically studied using a specially developed approach that includes a combination of the continuation method and the shooting technique. Using this approach, frequency responses of vibrations were obtained and bifurcations of periodic vibrations were studied.

Nonlinear vibrations of the structure are expanded into series by the eigenforms of linear vibrations. Therefore, the analysis of linear vibrations is an important step in the study of nonlinear dynamics of a structure. The Rayleigh-Ritz method was used to study linear vibrations.

Vibrations with significant amplitudes in the region of subharmonic resonance were detected, when the frequency of the excitation action is close to twice the natural frequency of the structure's vibrations. These subharmonic vibrations arise as a result of the period-doubling bifurcations of harmonic vibrations. Such subharmonic vibrations undergo saddle-node bifurcation and are polyharmonic with the predominant first harmonic of the Fourier series. Note that the period-doubling bifurcation has a single occurrence and does not lead to an infinite sequence of period-doubling bifurcations.

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Біфуркації та стійкість нелінійних коливань тришарової композитної оболонки з помірними амплітудами

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Авторами виведено математичну модель геометрично нелінійних коливань тришарових оболонок, яка описує коливання конструкції з амплітудами, порівняними з її товщиною. При виведенні цієї моделі використовується теорія зсуву високого порядку. Інерція обертання також враховується. При цьому середній шар є стільниковим заповнювачем, виготовленим завдяки адитивним технологіям FDM. Крім того, кожен шар оболонки описується п'ятьма змінними (трьома проєкціями переміщень і двома кутами повороту нормалі до серединної поверхні). Загальна кількість невідомих змінних дорівнює п'ятнадцяти. Для отримання моделі нелінійних коливань конструкції використано метод заданих форм. Виведено потенційну енергію, яка враховує квадратичні, кубічні й четверті степені узагальнених переміщень конструкції. Всі узагальнені переміщення розкладаються за узагальненими координатами і власними формами, які визнаються базовими функціями. Доведено, що математична модель коливань оболонки є системою нелінійних неавтономних звичайних диференціальних рівнянь. Для дослідження нелінійних періодичних коливань та їх біфуркацій застосовується чисельна процедура, яка є поєднанням методу продовження і методу пристрілювання. Метод пристрілювання враховує умови періодичності, що виражаються системою нелінійних рівнянь алгебри щодо початкових умов періодичних коли-

вань. Ці рівняння розв'язуються з використанням метода Ньютона. Чисельно досліджено властивості нелінійних періодичних коливань та їх біфуркацій в областях субгармонічних резонансів. Виявлено стійкі субгармонічні коливання другого порядку, які зазнають сідло-вузлової біфуркації. Нескінченної послідовності біфуркацій, що призводить до хаотичних коливань, не виявлено.

***Ключові слова**: оболонка подвійної кривизни, адитивні технології, стільниковий заповнювач, біфуркаційна поведінка.*

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