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CHAOTIC DYNAMICS OF CANTILEVER BEAMS WITH BREATHING CRACKS

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A nonlinear dynamic system with a finite number of degrees of freedom, which describes the forced oscillations of a beam with two breathing cracks, is obtained. The cracks are located on opposite sides of the beam. The Bubnov-Galerkin method is used to derive the nonlinear dynamic system. Infinite sequences of period-doubling bifurcations cause chaotic oscillations and are observed at the second-order subharmonic resonance. Poincaré sections and spectral densities are calculated to analyze the properties of chaotic oscillations. In addition, Lyapunov exponents are calculated to confirm the chaotic behavior. As follows from the numerical analysis, chaotic oscillations arise as a result of the nonlinear interaction between cracks.

***Keywords**: cracked beam, forced oscillations, period-doubling bifurcation, chaotic oscillations, Lyapunov exponent.*

Introduction

Oscillations of shafts and beams with cracks are a serious problem for the energy industry. For example, cracks appear on compressor blades and turbine shafts, which changes the dynamic characteristics of the structure. Oscillations of structures with cracks are mainly nonlinear [1]. Nonlinear oscillations of beams with cracks were analyzed by Christides and Barr [2]. They derived a partial differential equation of beam oscillations using the extended Hu-Washizu variational principle. In turn, Shen and Pierre [3] propose crack and displacement functions to describe oscillations of a beam with a crack. A system with one degree of freedom describing oscillations of a beam with a crack was obtained using the Christides and Barr beam theory [4, 5]. A detailed derivation of the partial differential equation describing oscillations of a beam with an open crack is given in [6]. Nonlinear shapes of a beam with cracks were considered by Chati, Rand and Mukherjee [7], and geometrically nonlinear oscillations of a beam with a breathing crack were considered by Carneiro and Ribeiro [8]. The crack is described by the delta function in the expression for the bending stiffness in [9]. Oscillations of a beam with an arbitrary number of cracks are studied by Caddemi, Cali and Marletta [10], and oscillations of a beam with geometric nonlinearity and an open crack are studied by Bikri, Benamar and Bennouna [11]. In [12], a nonlinear dynamic system is derived to describe oscillations of a rotor with a crack. A beam with several breathing cracks is studied by Sinha, Friswell, Edwards [13]. The natural frequencies and natural shapes of a beam with several open cracks are analyzed using the Euler-Bernoulli beam model by Stachowicz and Krawczuk [14]. The crack is taken into account by additional boundary conditions. Forced oscillations of beams with a transverse crack were studied by Plakhtienko and Yasinskii [15]. In [16], an asymptotic procedure based on the method of many scales was proposed for the analysis of forced oscillations of a beam with a breathing crack, and in [17], bifurcations of periodic oscillations of a beam with two cracks located on one side of the beam were numerically analyzed. The finite element method was used to calculate the nonlinear dynamics of beams with a breathing crack in [18–21]. The combination resonance of nonlinear oscillations of a beam is considered in [22]. The application of the Melnikov function to the analysis of subharmonic oscillations of a beam is studied in [23]. Flexural-flexural-torsional geometrically nonlinear oscillations of beams are studied in [24].

In this paper, a model of nonlinear oscillations of a beam with two breathing cracks located on opposite sides of the beam is derived. Two contact parameters are used to describe the crack breathing. A nonlinear dynamic system with a finite number of degrees of freedom is derived to describe the beam oscillations. Nonlinear oscillations in the region of the fundamental and second-order subharmonic resonance are analyzed using parameter extension. The properties of chaotic oscillations are considered. As follows from the numerical analysis, chaotic oscillations arise as a result of nonlinear interaction between cracks.

Problem statement and basic equations

Nonlinear bending oscillations of a cantilever beam with two breathing cracks are considered. The beam has a rectangular cross-section, and two breathing cracks are located on its opposite sides (Fig. 1).

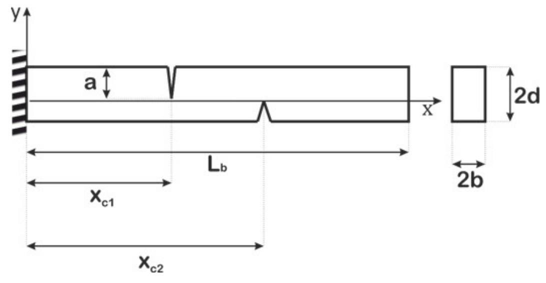

Fig. 1. Cantilever beam with two cracks

Each of the cracks can be opened or closed. If a crack is opened, the stiffness changes. To describe the nonlinear oscillations of the beam, a mathematical model that takes into account the local change in stiffness near the cracks is used. The crack is described by delta functions in the expression for the bending stiffness, and the oscillations of the beam with breathing cracks are described by a partial differential equation [17]

$$E_0 I_0 \left\{ [1-\sum k_i \gamma_i \delta(x-x_{ci})] \right\}''_{XX} + m\ddot{w} = p(x,t), \tag{1}$$

where $\frac{\partial^2 w}{\partial x^2} = w''_{XX}$; w is the transverse displacement of the beam; m is the mass per unit length; E_0I_0 is the bending stiffness of the beam without cracks; $p(x, t)$ is the external transverse force per unit length; $\delta(x–x_{ci})$ is the delta function; x_{ci} is the longitudinal coordinate of the crack; γ_i is the dimensionless damage intensity parameter [25]; k_i is the contact parameter. If the crack is open, then k_i=1, and if closed, then k_i=0.

According to [17], crack breathing is described by the sign $\frac{\partial^2 w}{\partial x^2}$. The condition for opening / closing of a crack located at a point ($x=x_{ci}$) takes the form

$$\begin{cases} w''\big|_{x=x_{c,1}} < 0;\ k_1 = 1; \\ w''\big|_{x=x_{c,1}} > 0;\ k_2 = 0. \end{cases}$$

Right crack opening/closing condition

$$\begin{cases} w''\big|_{x=x_{c,2}} > 0;\ k_1 = 1; \\ w''\big|_{x=x2} < 0;\ k_2 = 0. \end{cases}$$

The following four phases of beam motion are observed:

– the left crack is closed and the right one is open (k_1=0; k_2=1);

– the left and right cracks are closed (k_1=k_2=0);

– the left crack is open and the right one is closed (k_1=1; k_2=0);

– the left and right cracks are open (k_1=k_2=1).

The Bubnov–Galerkin method is applied to equation (1) to derive a nonlinear dynamic system with a finite number of degrees of freedom. In this case, the transverse displacements of the beam take the form

$$w(x,t) = \sum_{i-1}^{N_c} q_i(t)\Psi_i(x),$$

where $\Psi_i(x) = (1-k_1)k_2 W_{0,1}^{(i)}(x) + (1-k_1)(1-k_2) W_{0,0}^{(i)}(x) + k_1(1-k_2) W_{1,0}^{(i)}(x) + k_1 k_2 W_{1,1}^{(i)}(x)$; $W_{0,1}^{(i)}(x)$; $W_{0,0}^{(i)}(x)$; $W_{1,0}^{(i)}(x)$; $W_{1,1}^{(i)}(x)$ are eigenforms of the four phases of beam motion, considered above. Eigenforms $W_{k_1 k_2}^{(i)}$ are marked with contact parameters k_1 and k_2. For example, eigenform $W_{0,1}^{(i)}$ describes the oscillations of the beam when the left crack is closed and the right crack is open.

As a result of using the Bubnov–Galerkin method, a nonlinear dynamical system with a finite number of degrees of freedom is obtained

$$\sum_{i=1}^{N_c}\left[M_{ji}(q)\ddot{q}_i + R_{ji}(q)q_i\right] = \tilde{p}_j(t); \quad j=1, \ldots, N_c, \tag{2}$$

where $M_{ji}(q) = m\int_0^{L_b}\Psi_i(x)\Psi_j(x)dx$; $\tilde{p}_j(t) = \int_0^{L_b} p(x,t)\Psi_j(x)dx$; $R_{ji}(q) = E_0 I_0 \int_0^{L_b}\left\{\left[1-\sum_{i=1}^{2}k_i\gamma_i\delta(x-x_{ci})\right]\Psi_i''(x)\right\}''\Psi_j(x)dx$; $q=(q_1, \ldots, q_{N_c})$ is the generalized coordinate vector.

Matrix $R_{ji}(q)$ elements are calculated as follows

$$R_{ji}(q) = \tilde{\omega}_i^2 m\int_0^{L_b}\Psi_i(x)\Psi_j(x)dx,$$

where $\tilde{\omega}_i$ are natural frequencies at the corresponding phase of the structure's motions, which is determined by the contact parameter. Elements of matrix $R_{ji}(q)$ and matrix $M_{ji}(q)$ contain integrals $\int_0^{L_b}\Psi_i(x)\Psi_j(x)dx$, which depend on the phases of the structure's motion as

$$\int_0^{L_b}\Psi_i(x)\Psi_j(x)dx = \begin{cases} \int_0^{L_b} W_{1,1}^{(i)}(x)W_{1,1}^{(j)}(x)dx; \; G(x_{c1},q)<0; \; G(x_{c2},q)>0; \\ \int_0^{L_b} W_{1,0}^{(i)}(x)W_{1,0}^{(j)}(x)dx; \; G(x_{c1},q)<0; \; G(x_{c2},q)<0; \\ \int_0^{L_b} W_{0,1}^{(i)}(x)W_{0,1}^{(j)}(x)dx; \; G(x_{c1},q)>0; \; G(x_{c2},q)>0; \\ \int_0^{L_b} W_{0,0}^{(i)}(x)W_{0,0}^{(j)}(x)dx; \; G(x_{c1},q)>0; \; G(x_{c2},q)<0; \end{cases} \tag{3}$$

$$i=1, \ldots, N_c; \quad j=1, \ldots, N_c,$$

where $G(x,q) = \sum_{i=1}^{N_c}\frac{d^2W_{0,0}^{(i)}(x)}{dx^2}q_i(t)$.

The eigenforms of a beam with two closed cracks participate in relations (3). In this case, they enter equation (3) and satisfy the following orthogonality conditions:

$$\int_0^{L_b} W_{k_1k_2}^{(i)}W_{k_1k_2}^{(j)}d\xi = \pi_i^{(k_1k_2)}\delta_{ij}; \quad k_1=0, 1; \, k_2=0, 1; \, i=1, \ldots, N_c; \quad j=1, \ldots, N_c, \tag{4}$$

where δ_{ij} is the Kronecker symbol. Note that the orthogonality condition (4) satisfies all four phases of the structure's motions considered above.

Taking into account the internal friction of the beam material, the nonlinear system (2) takes on the following matrix form:

$$\begin{cases} M^{(1,1)}\ddot{q} + D\dot{q} + R^{(1,1)}q = F^{(1,1)}(t); \; G(x_{c1},q)<0; \; G(x_{c2},q)>0; \\ M^{(1,0)}\ddot{q} + D\dot{q} + R^{(1,0)}q = F^{(1,0)}(t); \; G(x_{c1},q)<0; \; G(x_{c2},q)<0; \\ M^{(0,1)}\ddot{q} + D\dot{q} + R^{(0,1)}q = F^{(0,1)}(t); \; G(x_{c1},q)>0; \; G(x_{c2},q)>0; \\ M^{(0,0)}\ddot{q} + D\dot{q} + R^{(0,0)}q = F^{(0,0)}(t); \; G(x_{c1},q)>0; \; G(x_{c2},q)<0, \end{cases} \tag{5}$$

where $F_j^{(1,1)}(t)=\dfrac{1}{m\pi_j^{(1,1)}}\tilde{p}_j(t)$; $F_j^{(1,0)}(t)=\dfrac{1}{m\pi_j^{(1,0)}}\tilde{p}_j(t)$; $F_j^{(0,1)}(t)=\dfrac{1}{m\pi_j^{(0,1)}}\tilde{p}_j(t)$; $F_j^{(0,0)}(t)=\dfrac{1}{m\pi_j^{(0,0)}}\tilde{p}_j(t)$.

Thus, the motions of the structure satisfy the ordinary differential equations (5).

The dynamic system (5) is rewritten in terms of the following dimensionless variables:

$$y=\frac{q}{2d};\ \tau=\omega_{0,0}^{(1)}t,$$

where $\omega_{0,0}^{(1)}$ is the first natural frequency of a beam without cracks; $2d$ is the height of the beam cross-section (Fig. 1). The dynamic system (5) with respect to dimensionless variables and parameters takes the form:

$$\begin{cases} M^{(1,1)}y''+\overline{D}y'+\overline{R}^{(1,1)}y=\overline{F}^{(1,1)}(\tau);\, G(x_{c1},y)<0;\, G(x_{c2},y)>0;\\ M^{(1,0)}y''+\overline{D}y'+\overline{R}^{(1,0)}y=\overline{F}^{(1,0)}(\tau);\, G(x_{c1},y)<0;\, G(x_{c2},y)<0;\\ M^{(0,1)}y''+\overline{D}y'+\overline{R}^{(0,1)}y=\overline{F}^{(0,1)}(\tau);\, G(x_{c1},y)>0;\, G(x_{c2},y)>0;\\ M^{(0,0)}y''+\overline{D}y'+\overline{R}^{(0,0)}y=\overline{F}^{(0,0)}(\tau);\, G(x_{c1},y)>0;\, G(x_{c2},y)<0, \end{cases} \tag{6}$$

where $y'=\dfrac{dy}{d\tau}$; $\overline{F}^{(i_1,i_2)}(t)=\dfrac{F^{(i_1,i_2)}(t)}{2d\omega_{0,0}^{(1)2}}$; $\overline{D}=\dfrac{D}{\omega_{0,0}^{(1)}}$; $\overline{R}^{(i_1,i_2)}(t)=\dfrac{R^{(i_1,i_2)}}{\omega_{0,0}^{(1)2}}$; i_1=1, 2; i_2=1, 2.

The frequency response of the nonlinear system (6) is analyzed further. The stability and bifurcations of periodic oscillations are numerically estimated. The parameter extension method is used to calculate the frequency responses. The theoretical justification of this method is given in [26, 27].

The system of variational equations derived from system (6) is obtained for the analysis of the stability and bifurcations of periodic oscillations [28]. Lyapunov exponents are calculated from numerical solutions of the variational equations. The stability and bifurcations of periodic motions are determined from the calculations of multipliers [28]. The results of numerical modeling of the stability and bifurcations of periodic motions are considered in the next paragraph.

Numerical analysis of nonlinear beam oscillations

The nonlinear oscillations of a cantilever beam with two breathing cracks were numerically analyzed (Fig. 1). The following numerical values of the geometric and physical parameters of the beam were selected: E=2.1×10^{11} Pa; ρ=7800 kg/m^3; L_b=0.117 m; x_c=0.015 m; b=0.005 m; d=0.005 m; a=0.8·d.

Forced oscillations are excited by periodic movements of the clamp

$$p(x,t)=H\Omega^2\sin\Omega t,$$

where the amplitude of periodic movements of the cantilever clamp is H=0.003·d. Two cracks have the following longitudinal coordinates x_{c1}=0.015 m; x_{c2}=0.5·L_b.

A numerical analysis of the nonlinear dynamic system (5) was performed. The parameters of periodic oscillations were numerically analyzed as functions Ω. To obtain these functions, the method of continuation by parameter [17, 26, 27] is used. The multipliers of periodic oscillations [28] are calculated to analyze their stability and bifurcations.

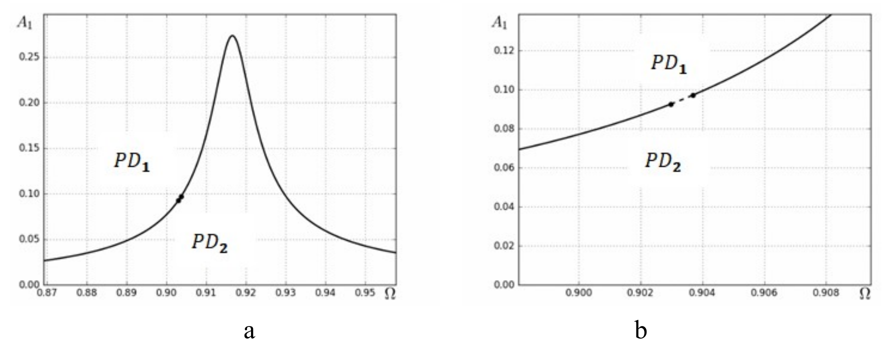

Fig. 2. Frequency response of a cantilever beam with two cracks

The frequency response of the main resonance is shown in Fig. 2, a. The amplitude dependence A_1 of oscillations $y_1(\tau)$ on frequency Ω is shown in Fig. 2, a. Two period-doubling bifurcations PD_1 and PD_2 are shown in Fig. 2, a. At these bifurcation points, stable periodic oscillations become unstable. The frequency response on a small scale is shown in Fig. 2, b. Stable and unstable oscillations are shown by solid and dashed curves, respectively.

Another period-doubling bifurcation is observed at a frequency of $\Omega = 1.75$. Then the oscillations with a period $T_1=2\pi/\Omega$ lose stability and oscillations with a period $T=2T_1$. These oscillations are shown in Fig. 3.

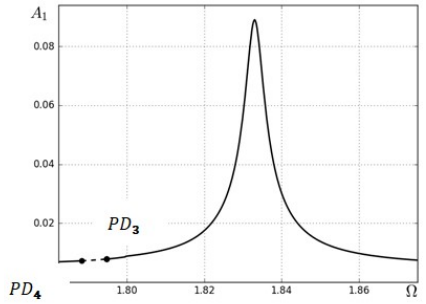

Fig. 3. Frequency response of second-order subharmonic oscillations

Second-order subharmonic oscillations (Fig. 3) undergo period-doubling bifurcations, which are denoted by PD_3 and PD_4.

Steady motions between the bifurcation points of the period-doubling are analyzed (Fig. 2 and Fig. 3). Direct numerical integration of the dynamical system (5) is performed. The oscillations of the system are considered as transient processes on the time interval $\tau \in [0; 1000 \cdot T_1]$. Poincaré sections [29] are calculated for the analysis of steady oscillations. After that, the transformation of the subspace of the dynamical system (6) into itself is analyzed. $y_1(\tau),..., y_{N_c}(\tau), \dot{y}_1(\tau),..., \dot{y}_{N_c}(\tau)$; $\frac{\tau}{T_1} \in Z$, where Z is the set of integers. This subspace is denoted as:

$$\Sigma = \left\{ (y_1,..., y_{N_c}, \dot{y}_1,...\dot{y}_{N_c}) \middle| \frac{\tau}{T_1} \in Z \right\}.$$

The Poincaré section defines the transformation in this way $\Sigma \to \Sigma$.

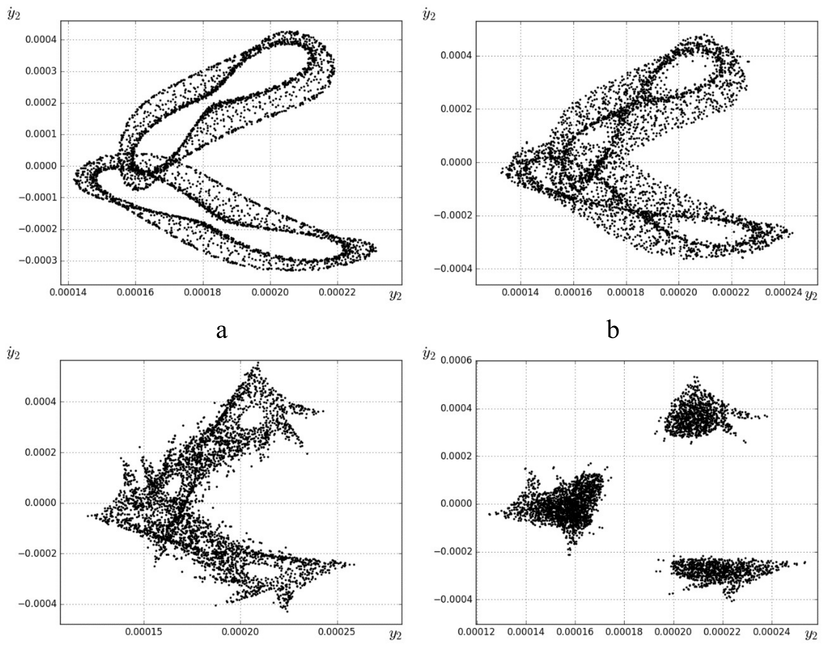

Fig. 4. Poincaré sections of chaotic oscillations at the following values of the perturbation frequency:
a – Ω=1.792; b – Ω=1.79225; c – Ω=1.7925; d – Ω=1.793

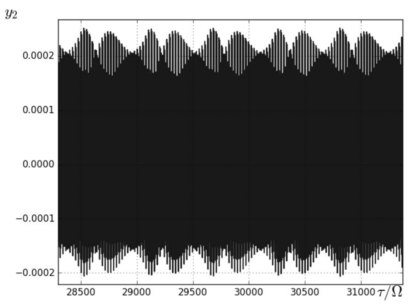

Fig. 5. Chaotic oscillations $y_2(\tau)$ at load frequency Ω=1,792

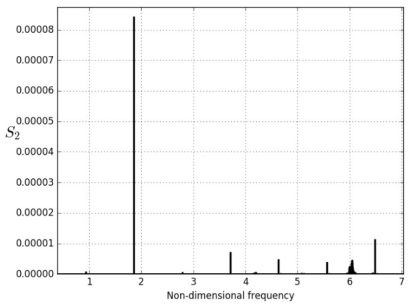

Fig. 6. Spectral densities $y_2(\tau)$ at load frequency Ω=1.7925

Steady oscillations between period-doubling bifurcations PD_1 and PD_2 are analyzed. These bifurcations lead to an infinite sequence of period-doubling bifurcations, resulting in the generation of chaotic oscillations. Fig. 4 shows the Poincaré cross sections of chaotic oscillations at frequencies: Ω=1.792; Ω=1.79225; Ω=1.7925; Ω=1.793. Chaotic oscillations $y_2(\tau)$ at Ω=1.792 are shown in Fig. 5.

The spectral densities of chaotic oscillations were numerically analyzed. The spectral density at Ω=1.7925 is shown in Fig. 6. Spectral densities consist of a set of discrete components, which are blurred when $\omega \approx 6$, which confirms the chaotic nature of the movement modes.

To analyze chaotic motions, the spectrum of Lyapunov exponents λ_i; i=1, 2, … as calculated. To calculate these values, the variational equations are solved numerically and Gram-Schmidt orthogonalization is performed. This procedure for calculating the Lyapunov spectrum is well known [28]. The behavior of the first Lyapunov exponent λ_1 in time at Ω=1.792 is shown in Fig. 7. As follows from this figure, the first Lyapunov exponent has a positive value. Therefore, the motion of the system at Ω=1.792 is chaotic.

The Lyapunov spectrum was calculated for different values of the load frequencies Ω. The first four Lyapunov exponents are given in Table 1. Thus, they are positive, which confirms the chaotic nature of the oscillations.

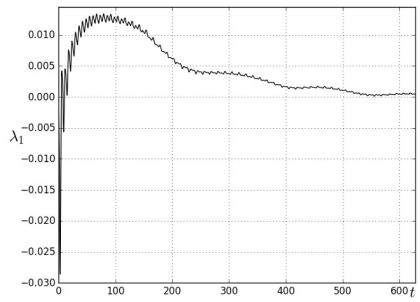

Fig. 7. Behavior of the first Lyapunov exponent in time at Ω=1.792

Table 1. Lyapunov spectrum

Ω	λ_1	λ_2	λ_3	λ_4
1.79200	4.229×10^{-4}	-3.107×10^{-3}	-6.033×10^{-3}	-8.688×10^{-3}
1.79225	4.201×10^{-4}	-3.324×10^{-3}	-5.999×10^{-3}	-8.804×10^{-3}
1.79250	4.339×10^{-4}	-4.142×10^{-3}	-6.013×10^{-3}	-9.720×10^{-3}
1.79300	4.126×10^{-4}	-3.198×10^{-3}	-5.989×10^{-3}	-8.775×10^{-3}

Conclusions

Forced oscillations of a beam with two cracks located on opposite sides of the beam are described by a nonlinear dynamical system with finite degrees of freedom. This nonlinear dynamical system is obtained by applying the Bubnov–Galerkin method to the nonlinear partial differential equation of motion of the system.

The motions of the beam consist of the following four phases: the left crack is closed and the right one is open; the left and right cracks are closed; the left crack is open and the right one is closed; the left and right cracks are open. During one oscillation period, four phases of the system motions occur sequentially. The nonlinear properties of the system are realized by transitions between the four phases.

In this cracked beam, infinite sequences of period-doubling bifurcations and chaotic motions arise. The sequence of period-doubling bifurcations and chaotic motions are observed at subharmonic resonance.

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Хаотична динаміка консольних балок із дихаючими тріщинами

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Отримано нелінійну динамічну систему зі скінченним числом ступенів свободи, яка описує вимушені коливання балки з двома дихаючими тріщинами. Тріщини розташовані на протилежних сторонах балки. Для виведення нелінійної динамічної системи застосовано метод Бубнова-Гальоркіна. Нескінченні послідовності біфуркацій подвоєння періоду викликають хаотичні коливання і спостерігаються при субгармонічному резонансі другого порядку. Для аналізу властивостей хаотичних коливань розраховано перерізи Пуанкаре і спектральні щільності. Крім того, показники Ляпунова розраховуються для підтвердження хаотичної поведінки. Як випливає з чисельного аналізу, хаотичні коливання виникають внаслідок нелінійної взаємодії між тріщинами.

Ключові слова: *балка з тріщинами, вимушені коливання, біфуркація подвоєння періоду, хаотичні коливання, показник Ляпунова.*

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