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FIRST MAIN PROBLEM OF THE THEORY OF ELASTICITY FOR A LAYER WITH TWO THICK-WALLED PIPES AND ONE CYLINDRICAL CAVITY

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Structures that are fixed on cylindrical inclusions are among the most common ones in machine and aircraft construction. Some of these inclusions can be modeled as thick-walled pipes with specified stress values on the inner surface. However, the literature does not provide accurate methods for calculating these structures, which indicates the relevance of posing and solving such problems. The presented paper considers the solution method for the model of the structure, which is an elastic homogeneous layer located on two pipes embedded into it and having a longitudinal cylindrical cavity that is parallel to layer boundaries. On the flat surfaces of the cavity surface layer, on the inner surfaces of the pipes, the stresses are considered known. When solving the problem, two types of coordinate systems are used: Cartesian for the layer and cylindrical for the pipes and cavity. The basic solutions in different coordinate systems are given as Lamé equations and combined using transition functions and the generalized Fourier method. An infinite system of integro-alberic equations is formed based on the boundary conditions on the upper and lower surfaces of the layer, the surface of the cavity, and the continuity conditions between the layer and the pipes. After that, the system of equations is reduced to linear algebraic equations of the second kind, to which the reduction method is applied. The problem is solved numerically with a given accuracy, which allowed obtaining the stress-strain state at any point of the elastic body. An analysis of the stress state is carried out with different values of the distance between the thick-walled pipes. On the upper and lower boundaries of the layer, and on the surface of the cylindrical surface, the stresses are considered known. The obtained results do not show a significant effect on the stress along the lower and upper surfaces of the layer. At the same time, the stresses in the layer along the surface of the pipe and layer junction decrease as the distance between the pipes increases. The obtained numerical results can be used in the prediction of geometric parameters during design.

Keywords: *layer with cylindrical inclusions, thick-walled pipes, generalized Fourier method, Lamé equation, fibrous composite.*

Introduction

At the stage of designing aircraft and machine-building structures, the problem of choosing a calculation scheme is important to solve. In this case, the zones of body junctions, depending on the mutual relations of sizes and mechanical characteristics, can be modeled in calculation models in the form of cavities or various kinds of inclusions. One of the options for simplifying the calculation scheme is to model the smaller of the bodies - a thick-walled pipe with mechanical properties different from those of the main body.

The choice of the calculation method by which the stress state in the body will be determined is no less important due to the fact that the correctness of this choice directly affects the accuracy of the obtained calculation results.

As of now, the most common practice is the use of structural mechanics methods or various kinds of numerical methods to solve such problems [1, 2]. An example of this type of research is the paper [3], in which the finite element method is used to analyze the stress state. A solution for a half-space reinforced by a plate with a vertical cylindrical cavity strengthened with a shell is given in this paper. The disadvantage of structural mechanics methods is the significant simplification of the model during the calculation, while numerical methods are approximate and do not take into account the infinite boundaries of the body. The abovementioned disadvantages lead to the fact that the use of structural mechanics methods or numerical methods cannot guarantee high accuracy of the final result [4].

To obtain accurate results, analytical methods must be used [5, 6], but, unfortunately, they cannot take into account more than three spatial boundary surfaces.

In addition, there are a large number of papers in which the cylindrical cavity or inclusions are perpendicular to the surfaces of the layer [7–11]. The solution for the wave field of an infinite elastic layer, which is weakened by a cylindrical cavity, was found in [7]. The conditions of fixation are given in the form of ideal contact along the upper and lower surfaces of the layer. The loads applied to the layer are given in the form of a tensile load acting along the cylindrical cavity. The problem statement in [8] is similar, taking into account that the lower surface of the layer has a rigid fixation. To obtain the results in [7, 8], the Laplace integral transforms and the integral Fourier method were useful. These methods were applied to the boundary conditions and axisymmetric equations of motion, which create a one-dimensional vector inhomogeneous boundary value problem. The main drawback of the abovementioned approach is that it does not allow to obtain a solution to the problem with several boundary surfaces.

In [9], an analytical solution was found for functionally graded (FG) composite laminated plates with circular cutouts under different loading conditions, and the analytically obtained results and the results obtained using the complex variable method and the finite element method were compared.

The problem of finding the distribution of thermal stresses in symmetric composite plates with non-circular holes under the action of a uniform heat flux was solved in [10]. At the same time, the problem was first solved for laminated composite plates with circular holes, and then, using the mapping function, the results for the plate with non-circular holes were obtained. This approach does not guarantee the accuracy of the final results due to the fact that the used methods are approximate.

Paper [11] is devoted to obtaining the distribution of stresses in an elastic half-space with a vertical cylindrical cavity when it is loaded using a coaxial stamp rotating under the action of a torque around its own axis. The solution was obtained using Weber-Orr integral transforms.

Unfortunately, papers [7–11] are focused on solving problems with the location of a cavity or inclusion perpendicular to the surfaces of the layer, so the results obtained in them cannot be used to find a solution to the problem for a layer containing inclusions located parallel to its surfaces without significant refinement.

The heterogeneity of the layer model, with inclusions in the form of tubes and cavities, can be taken into account with methods used for calculating composite materials [12–15].

Paper [12] is devoted to determining the dynamic stress state for two rods of different lengths, connected in an overlap, under the action of a longitudinal force applied to one of the rods.

The behavior of multilayer structures under the action of a dynamic load arising from a transverse impact was studied in paper [13]. The solution was obtained using the theory of a two-dimensional discrete structure. In the process of solving, the displacement function for each of the layers was given in the form of a power series. The theoretically obtained results were verified by experimental studies.

The study of the stress state in laminated aircraft glass units is devoted to papers [14] and [15]. Thus, in paper [14] the thermal stress state in a laminated glass unit, which is considered as an open cylindrical multilayer shell of constant thickness, was studied. It is believed that thermal loads arose under the action of interlayer film heat sources. An analytical solution is provided in the paper. A method for assessing the strength of a laminated glass unit in a collision with a bird was proposed in paper [15]. When solving the problem, the reduction in thickness and rotational inertia of the element of each layer was taken into account. The results of papers [12–15] also cannot be used to solve the problem posed in this paper without very significant refinement.

Thus, to obtain a solution to the problem with a predetermined accuracy, the most promising is the analytical-numerical generalized Fourier method [16]. It allows obtaining a solution for models consisting of a group of bodies, each of which has its own coordinate system, and it is possible to use several types of coordinate systems simultaneously.

Using the generalized Fourier method, solutions for an elastic cylinder with cylindrical cavities were obtained in papers [17, 18], with cylindrical inclusions - in paper [19], and for a half-space with a spheroidal cavity – in paper [20]. In this case, local cylindrical coordinate systems and formulas for the transition of basic solutions between them were used, and for a layer with a spheroidal cavity - Cartesian and spherical coordinate systems.

In papers [21–23], formulas of various types are proposed for the transition of basic solutions between cylindrical and Cartesian coordinate systems: in paper [21] – for a half-space with a cylindrical cavity; [22] – for a layer with a cavity on the surface of which stresses are given; in [23], a solution for a layer with a cylindrical inclusion, for which the displacements are assumed to be known, is given. However, the formulas for the transition between local coordinate systems necessary to obtain results for a layer with several inhomogeneities are not applied in these papers.

Papers [24–26] are devoted to increasing the number of bodies taken into account in the computational model. Thus, in paper [24], a situation is considered when Cartesian coordinate systems are used for two of the bodies (layer and half-space), and the origin of the third one – cylindrical coordinate system – coincides with the origin of the half-space coordinates. In [25], a layer fixed on two cylindrical supports was studied, and [26] was devoted to the study of the stress state in a layer with two continuous cylindrical inclusions and mixed boundary conditions.

The papers [27, 28] are devoted to the determination of the stress state in models where the inhomogeneity is given in the form of pipes. However, they take into account only one inhomogeneity, and there are no transition formulas between shifted cylindrical coordinate systems.

The abovementioned transition formulas are given in [29], which is devoted to the analysis of the stress state of a layer with two swivel joints and a cylindrical cavity. The presented paper is a continuation of the research, which begun in [29], while the swivel joints are replaced by thick-walled pipes, which requires the introduction of additional continuity conditions.

Problem statement

The elastic homogeneous layer is located on two pipes embedded into it and has a longitudinal cylindrical cavity parallel to its boundaries (Fig. 1).

The pipes are considered in a cylindrical coordinate system, and their geometric dimensions are given by the outer radius R_p, and the inner radius r_p, where p is the number of cylindrical inhomogeneity. The layer is given in the Cartesian coordinate system (x, y, z), the cavities are given in local cylindrical coordinate systems (ρ_p, φ_p, z). The upper boundary of the layer is a plane with a constant coordinate $y=h$, the lower boundary is the same with $y=-\tilde{h}$.

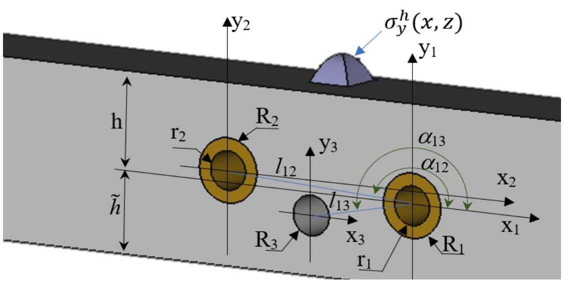

Fig. 1. Layer with two thick-walled pipes and a cylindrical cavity embedded into it

It is necessary to find a solution to the Lamé equation $\Delta\vec{u} + (1-2\sigma)^{-1}\nabla div\vec{u} = 0$.

The stresses at the upper and lower boundaries of the layer and on the surface of the cylindrical cavity (p=3) are given, respectively

$$F\vec{U}(x,z)\big|_{y=h} = \vec{F}_h^0(x,z); \quad F\vec{U}(x,z)\big|_{y=-\tilde{h}} = \vec{F}_{\tilde{h}}^0(x,z); \quad F\vec{U}(\varphi_3,z)\big|_{\rho_3=R_3} = \vec{F}_{\tilde{h}}^0(\varphi_3,z). \tag{1}$$

where $\vec{U}$ is the displacement in the layer; $F\vec{U} = 2\cdot G\cdot\left[\dfrac{\sigma}{1-2\cdot\sigma}\vec{n}\cdot \mathrm{div}\vec{U} + \dfrac{\partial}{\partial n}\vec{U} + \dfrac{1}{2}\left(\vec{n}\times \mathrm{rot}\vec{U}\right)\right]$ is the stress operator.

Stress distribution functions

$$\vec{F}_h^0(x,z) = \tau_{yx}^{(h)}(x,z)\cdot\vec{e}_x + \sigma_y^{(h)}(x,z)\cdot\vec{e}_y + \tau_{yz}^{(h)}(x,z)\cdot\vec{e}_z,$$

$$\vec{F}_{\tilde{h}}^0(x,z) = \tau_{yx}^{(\tilde{h})}(x,z)\cdot\vec{e}_x + \sigma_y^{(\tilde{h})}(x,z)\cdot\vec{e}_y + \tau_{yz}^{(\tilde{h})}(x,z)\cdot\vec{e}_z,$$

$$\vec{F}_R^0(\varphi_3,z) = \sigma_\rho^{(3)}(x,z)\cdot\vec{e}_\rho + \tau_{\rho\varphi}^{(3)}(x,z)\cdot\vec{e}_\varphi + \tau_{\rho z}^{(3)}(x,z)\cdot\vec{e}_z$$

are considered known.

On the inner surfaces of the pipes p=1, p=2 the normal and tangential stresses are also given

$$F\vec{U}(\varphi_1,z)\big|_{\rho_1=r_1} = \vec{F}_1^0(\varphi_1,z), \; F\vec{U}(\varphi_2,z)\big|_{\rho_2=r_2} = \vec{F}_2^0(\varphi,z). \tag{2}$$

Continuity conditions – equality of displacements and stresses along the contacting surfaces of each of the pipes and the layer

$$\vec{U}_0(\varphi,z)\big|_{\rho=R_1} = \vec{U}_p(\varphi,z)\big|_{\rho=R_1}; \quad \vec{U}_0(\varphi,z)\big|_{\rho=R_2} = \vec{U}_p(\varphi,z)\big|_{\rho=R_2}; \tag{3}$$

$$F\vec{U}_0(\varphi,z)\big|_{\rho=R_1} = F\vec{U}_p(\varphi,z)\big|_{\rho=R_1}; \quad F\vec{U}_0(\varphi,z)\big|_{\rho=R_2} = F\vec{U}_p(\varphi,z)\big|_{\rho=R_2}. \tag{4}$$

We will assume that as the distance from the origin increases along the z-axis and x-axis, all given functions asymptotically approach zero.

Problem solving

To solve the problem, the displacement in the layer is given in the form [17]

$$\vec{U}_0=\sum_{k=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(H_k(\lambda,\mu)\cdot\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)+\widetilde{H}_k(\lambda,\mu)\cdot\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)\right)d\mu\cdot d\lambda+$$

$$+\sum_{p=1}^{3}\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}B_{k,m}^{(p)}(\lambda)\vec{S}_{k,m}\left(\rho_p,\varphi_p,z;\lambda\right)d\lambda\,. \tag{5}$$

Displacement in the pipes in the form of [20]

$$\vec{U}_1=\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{k,m}^{(1)}(\lambda)\cdot\vec{R}_{k,m}(\rho_1,\varphi_1,z;\lambda)+\widetilde{A}_{k,m}^{(1)}(\lambda)\cdot\vec{S}_{k,m}(\rho_1,\varphi_1,z;\lambda)d\lambda,$$

$$\vec{U}_2=\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{k,m}^{(2)}(\lambda)\cdot\vec{R}_{k,m}(\rho_2,\varphi_2,z;\lambda)+\widetilde{A}_{k,m}^{(2)}(\lambda)\cdot\vec{S}_{k,m}(\rho_2,\varphi_2,z;\lambda)d\lambda,$$

where $H_k(\lambda,\mu)$, $\widetilde{H}_k(\lambda,\mu)$, $B_{k,m}^{(p)}(\lambda)$, $A_{k,m}^{(1)}(\lambda)$, $\widetilde{A}_{k,m}^{(1)}(\lambda)$, $A_{k,m}^{(2)}(\lambda)$, $\widetilde{A}_{k,m}^{(2)}(\lambda)$ are unknown functions found from boundary conditions (1), (2) and continuity conditions (3), (4).

Basic solutions of the Lamé equation $\vec{S}_{k,m}\left(\rho_p,\varphi_p,z;\lambda\right)$, $\vec{R}_{k,m}\left(\rho_p,\varphi_p,z;\lambda\right)$, $\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)$, $\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)$ are given in the form [27]

$$\vec{u}_k^{\pm}(x,y,z;\lambda,\mu)=N_k^{(d)}e^{i(\lambda z+\mu x)\pm\gamma y};$$
$$\vec{R}_{k,m}(\rho,\varphi,z;\lambda)=N_k^{(p)}I_m(\lambda\rho)e^{i(\lambda z+m\varphi)};$$
$$\vec{S}_{k,m}(\rho,\varphi,z;\lambda)=N_k^{(p)}\left[(\operatorname{sign}\lambda)^m K_m(|\lambda|\rho)\cdot e^{i(\lambda z+m\varphi)}\right];\quad k=1,2,3;$$

$$N_1^{(d)}=\frac{1}{\lambda}\nabla;\quad N_2^{(d)}=\frac{4}{\lambda}(\nu-1)\,\vec{e}_2^{(1)}+\frac{1}{\lambda}\nabla(y\,\cdot);\quad N_3^{(d)}=\frac{i}{\lambda}\operatorname{rot}\left(\vec{e}_3^{(1)}\cdot\right);$$

$$N_1^{(p)}=\frac{1}{\lambda}\nabla;\quad N_2^{(p)}=\frac{1}{\lambda}\left[\nabla\left(\rho\frac{\partial}{\partial\rho}\right)+4(\nu-1)\left(\nabla-\vec{e}_3^{(2)}\frac{\partial}{\partial z}\right)\right];$$

$$N_3^{(p)}=\frac{i}{\lambda}\operatorname{rot}\left(\vec{e}_3^{(2)}\right);\quad\gamma=\sqrt{\lambda^2+\mu^2}\,;\quad-\infty<\lambda,\mu<\infty\,,$$

where σ is the Poisson's ratio; $I_m(x)$, $K_m(x)$ are modified Bessel functions.

The infinite system of integro-algebraic equations has 9 unknowns and consists of five equations satisfying the boundary conditions (1) and (2) and four continuity conditions (3) and (4). Due to the fact that the components of equations (5) are written in a different coordinate system, the transition formulas between the basic solutions were used [26]:

– to switch from basic solutions $\vec{S}_{k,m}$ of cylindrical coordinate system to the solutions of the layer $\vec{u}_k^{(-)}$ (at $y>0$) and $\vec{u}_k^{(+)}$ (at $y<0$)

$$\begin{aligned}&\vec{S}_{k,m}\left(\rho_p,\varphi_p,z;\lambda\right)=\frac{(-i)^m}{2}\int_{-\infty}^{\infty}\omega_{\mp}^{m}\cdot e^{-i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\vec{u}_k^{(\mp)}\cdot\frac{d\mu}{\gamma},\quad k=1,\ 3;\\&\vec{S}_{2,m}\left(\rho_p,\varphi_p,z;\lambda\right)=\frac{(-i)^m}{2}\int_{-\infty}^{\infty}\omega_{\mp}^{m}\cdot\left(\left(\pm m\cdot\mu-\frac{\lambda^2}{\gamma}\pm\lambda^2\bar{y}_p\right)\vec{u}_1^{(\mp)}\mp\lambda^2\vec{u}_2^{(\mp)}\pm4\mu(1-\sigma)\vec{u}_3^{(\mp)}\right)\cdot\frac{e^{-i\mu\bar{x}_p\pm\gamma\bar{y}_p}d\mu}{\gamma^2},\end{aligned}\tag{6}$$

where $\gamma=\sqrt{\lambda^2+\mu^2}$; $\omega_{\mp}(\lambda,\mu)=\frac{\mu\mp\gamma}{\lambda}$; $m=0,\pm1,\pm2,\ldots$;

– to switch from basic solutions $\vec{u}_k^{(+)}$ and $\vec{u}_k^{(-)}$ of the layer to solutions $\vec{R}_{k,m}$ of the cylindrical coordinate system

$$\vec{u}_k^{(\pm)}(x,y,z)=e^{i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\sum_{m=-\infty}^{\infty}(i\cdot\omega_{\mp})^m\vec{R}_{k,m},\ (k=1,\ 3);$$
$$\vec{u}_2^{(\pm)}(x,y,z)=e^{i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\sum_{m=-\infty}^{\infty}\left[(i\cdot\omega_{\mp})^m\cdot\lambda^{-2}\left(\left(m\cdot\mu+\bar{y}_p\cdot\lambda^2\right)\cdot\vec{R}_{1,m}\pm\gamma\cdot\vec{R}_{2,m}+4\mu(1-\sigma)\vec{R}_{3,m}\right)\right],\tag{7}$$

where $\vec{R}_{k,m}=\vec{\tilde{b}}_{k,m}(\rho_p,\lambda)\cdot e^{i(m\varphi_p+\lambda z)}$;

$$\vec{\tilde{b}}_{1,n}(\rho,\lambda)=\vec{e}_\rho\cdot I_n'(\lambda\rho)+i\cdot I_n(\lambda\rho)\cdot\left(\vec{e}_\varphi\frac{n}{\lambda\rho}+\vec{e}_z\right);$$

$$\vec{\tilde{b}}_{2,n}(\rho,\lambda)=\vec{e}_\rho\cdot\left[(4\sigma-3)\cdot I_n'(\lambda\rho)+\lambda\rho I_n''(\lambda\rho)\right]+\vec{e}_\varphi i\cdot m\left(I_n'(\lambda\rho)+\frac{4(\sigma-1)}{\lambda\rho}I_n(\lambda\rho)\right)+\vec{e}_z i\lambda\rho I_n'(\lambda\rho);$$

$$\vec{\tilde{b}}_{3,n}(\rho,\lambda)=-\left[\vec{e}_\rho\cdot I_n(\lambda\rho)\frac{n}{\lambda\rho}+\vec{e}_\varphi\cdot i\cdot I_n'(\lambda\rho)\right];$$

$\vec{e}_\rho$, $\vec{e}_\varphi$, $\vec{e}_z$ are unit vectors in a cylindrical coordinate system;

– to switch from the basic solutions of the cylinder with number p to the solutions of the cylinder with number q

$$\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)=\sum_{n=-\infty}^{\infty}\vec{b}_{k,pq}^{mn}(\rho_q)\cdot e^{i(n\varphi_q+\lambda z)},k=1,\ 2,\ 3;$$
$$\vec{b}_{1,pq}^{mn}(\rho_q)=(-1)^n\tilde{K}_{m-n}(\lambda\ell_{pq})\cdot e^{i(m-n)\alpha_{pq}}\cdot\vec{\tilde{b}}_{1,n}(\rho_q,\lambda);$$
$$\vec{b}_{2,pq}^{mn}(\rho_q)=(-1)^n\left\{\tilde{K}_{m-n}(\lambda\ell_{pq})\cdot\vec{\tilde{b}}_{2,n}(\rho_q,\lambda)-\frac{\lambda}{2}\ell_{pq}\cdot\left[\tilde{K}_{m-n+1}(\lambda\ell_{pq})+\tilde{K}_{m-n-1}(\lambda\ell_{pq})\right]\cdot\vec{\tilde{b}}_{1,n}(\rho_q,\lambda)\right\}\cdot e^{i(m-n)\alpha_{pq}},$$
$$\vec{b}_{3,pq}^{mn}(\rho_q)=(-1)^n\tilde{K}_{m-n}(\lambda\ell_{pq})\cdot e^{i(m-n)\alpha_{pq}}\cdot\vec{\tilde{b}}_{3,n}(\rho_q,\lambda),\tag{8}$$

where α_{pq} is the angle between the x_p axis and the segment ℓ_{qp}, $\tilde{K}_m(x)=(sign(x))^m\cdot K_m(|x|)$.

After using the formulas for the transition of basic solutions between coordinate systems (6)–(8), the system of equations was given in one coordinate system. Thus, the infinite integro-algebraic system of equations was reduced to an infinite linear system of equations, to which the reduction method was applied [22]. The order of the system of equations m is a parameter of the accuracy of the calculation results.

Numerical studies of the stressed state

Two homogeneous thick-walled pipes pass through an elastic isotropic layer (Fig. 1). In addition, the layer has one cylindrical cavity. Poisson's ratio of the layer (D16T alloy) σ=0.3; modulus of elasticity E=71000 N/mm^2. Poisson's ratio of pipes (ShKh15 steel) σ=0.28, modulus of elasticity E=216000 N/mm^2.

Geometric parameters of the model: outer radius of pipes R_1=R_2=16 mm, inner one r_1=r_2=11 mm, cavity radius R_2=16 mm, distance to the upper and lower boundaries of the layer h=32 mm, $\tilde{h}$ =22 mm. The pipes and the cavity are arranged parallel to each other, with their central axes lying on a horizontal plane parallel to the upper and lower boundaries of the layer, thus, α_{12}=0, α_{13}=π. The calculation was performed with different distances between the pipes L_{12}=80 mm and L_{12}=100 mm.

Normal stresses are given at the upper boundary of the layer in the form of a unit wave $\sigma_y^{(h)}(x,z)=-10^8\cdot(z^2+10^2)^{-2}\cdot(x^2+10^2)^{-2}$ and zero tangential stresses $\tau_{yx}^{(h)}=\tau_{yz}^{(h)}=0$; at the lower boundary – normal stresses in the form of a unit wave and zero tangential stresses so that the layer is in equilibrium. Tangential stresses at the lower boundary of the layer are zero. Normal stresses on the inner surfaces of the pipes and the cavity are also zero.

The infinite system was truncated by the parameter m=4 (the number of terms of the Fourier series and the order of the system of equations).

The integrals were calculated using Filon quadrature formulas. The accuracy of meeting the boundary conditions at the specified m and the specified geometric parameters is not less than 10^{-5} at values from 0 to 1. The obtained results are shown in Figs. 2–5.

Fig. 2 shows graphs of the specified stresses σ_y and the corresponding stresses σ_x on the upper surface of the layer at z=0. The results of calculations with different distances between the pipes are shown: L_{12}=80 mm and L_{12}=100 mm. On the lower surface of the layer, the stress distribution diagram and values remain the same.

As the graphs in Fig. 2 show, changing the distance between the pipes does not significantly affect the stress distribution on the surface of the layer. Moreover, the stress distribution σ_x is very similar to the distribution of the given stresses σ_y, except that the stress graph σ_x demonstrates the presence of both compressed and stretched zones in the layer. With increasing distance from the location of the extremum on the diagram of the given stresses σ_y, the stress values decrease, and their graph asymptotically approaches zero.

Fig. 3 shows the stress σ_ρ in the layer along the continuity surface of the right pipe and the layer (p=1) at z=0.

The maximum stresses σ_ρ occur along the outer lateral side of the pipe and have a negative sign (Fig. 3). Along the inner lateral surface of the pipe, the stresses are positive and much smaller in magnitude. In addition, it should be noted that the maximum values of the stresses σ_ρ are inversely proportional to the distance between the pipes.

Fig. 4. shows a graph of the stresses σ_φ along the continuity surface of the right pipe and the layer (p=1) at z=0.

These graphs also show an inverse proportionality for the stress value and the distance between the pipes.

The most interesting results are given in Fig. 5, which shows the stress graph σ_z in the layer along the continuity surface at z=0.

As shown in this graph (Fig. 5), as the distance between the pipes increases, the values on the σ_z diagram become more uniform and closer to the average value.

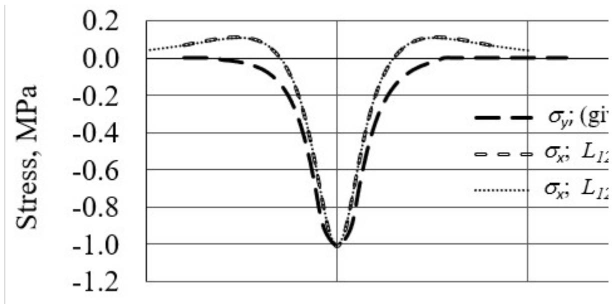

Fig. 2. Stress on the upper surface of the layer

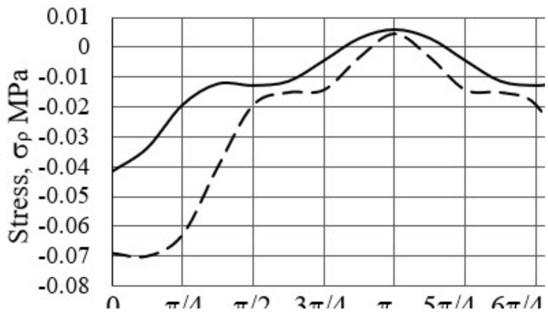

Fig. 3. Stress σ_ρ along the continuity surface of the pipe and layer p=1

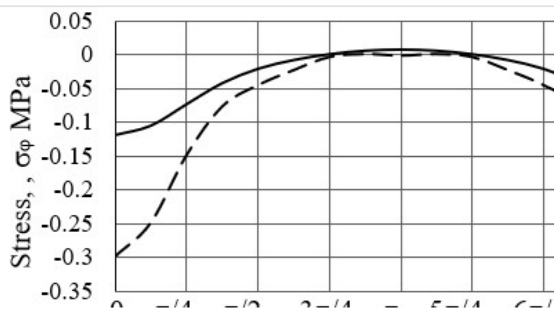

Fig. 4. Stress σ_φ along the continuity surface of the pipe and layer p=1

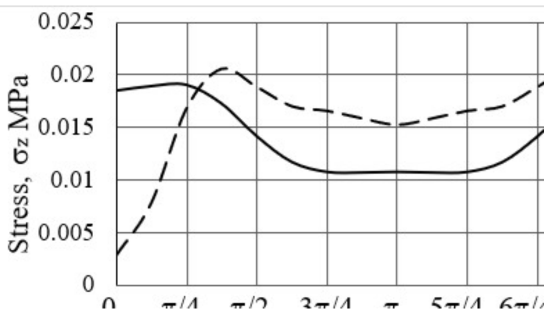

Fig. 5. Stress σ_z along the continuity surface of the pipe and layer p=1

Conclusions

A new problem is solved for a layer with a longitudinal cylindrical cavity (parallel to its boundaries) and located on two thick-walled pipes embedded into it.

The pipes are given in the form of bodies for which the continuity conditions along the surface where they contact the layer are given. This allowed reducing the problem to the classical model of the spatial theory of elasticity. Its solution was performed using the analytical-numerical generalized Fourier method, which made it possible to obtain a solution with a given accuracy. The stress state analysis showed the distribution of internal stresses in the layer and pipes.

The obtained results indicate that with an increase in the distance between the pipes, the values of internal stresses in the layer along the continuity surface of the layer and pipes decrease.

The solution method proposed in the paper can be applied to a larger number of pipes and cavities.

Further development of this paper is possible in the direction of complicating the mathematical model by increasing the number of bodies made of different materials and constructing models from several pipes nested one inside the other.

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Перша основна задача теорії пружності для шару з двома товстостінними трубами й однією циліндричною порожниною

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Конструкції, закріплені на циліндричних включеннях, є серед найпоширеніших у машино- й авіабудуванні. Певна кількість таких включень може бути промодельована в розрахункових моделях як товстостінні труби для яких задано значення напружень на внутрішній поверхні. Однак у літературі не наведено точних методів розрахунку вищезгаданих конструкцій, що свідчить про актуальність постановки і вирішення таких завдань. У поданій роботі розглянуто метод розв'язання для моделі конструкції, яка представлена у вигляді пружного однорідного шару, розташованого на двох врізаних у нього трубах, і має поздовжню циліндричну порожнину, паралельну його межам. На плоских поверхнях шару поверхні порожнини, на внутрішніх поверхнях труб напруження вважаються відомими. При розв'язанні задачі застосовано системи координат двох типів: декартова для шару й циліндричні – для труб і порожнини. Базові розв'язки в різних системах координат представлені у вигляді рівнянь Ламе і поєднані за допомогою функцій переходу узагальненого методу Фур'є. Нескінчена система інтегро-алебраїчних рівнянь сформована, спираючись на граничні умови на верхній та нижній поверхнях шару, поверхні порожнини й умови спряження між шаром і трубами. Після цього система рівнянь була зведена до лінійних алгебраїчних рівнянь другого роду, до яких застосовано метод редукції. Задача розв'язана чисельно із наперед заданою точністю, що дозволило отримати характеристики напруженого стану у будь-якій точці пружного тіла. Проведено аналіз напруженого стану з різними значеннями відстані між товстостінними трубами. На верхній та нижній межах шару, на поверхні циліндричної порожнини напруження вважаються відомими. Отримано результати, які не показали суттєвого впливу відстані між товстостінними трубами на напруження уздовж нижньої та верхньої поверхонь шару. При цьому напруження в шарі вздовж поверхні спряження труби й шару при збільшенні відстані між трубами зменшуються. Отримано числові результати, що можуть бути застосовані при прогнозуванні геометричних параметрів під час проєктування конструкції, які закріплені за допомогою циліндричних включень.

***Ключові слова**: шар з циліндричними включеннями, товстостінні труби, узагальнений метод Фур'є.*

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