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STUDY OF THE STRESS STATE OF SOLID CYLINDERS WITH INHOMOGENEOUS STRUCTURE UNDER VARIOUS BOUNDARY CONDITIONS AT THE ENDS

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Solving the problems of the theory of elasticity on the stress state of continuous-inhomogeneous bodies requires the improvement of existing and the development of new numerical-analytical methods. This is due to the need to fully take into account arbitrary dependencies of the mechanical properties of the material on the coordinates and nature of the applied load. The paper is devoted to the solution of the axisymmetric problem of the linear theory of elasticity on the equilibrium of the solid inhomogeneous cylinders of the finite length with the different boundary conditions at the ends. The polymeric continuous-inhomogeneous material with a gradient profile corresponding to the quadratic variants of change of the Young's modulus along the radial coordinate is considered. The solution of the problem is based on the application of the method of spline-approximation of functions in the direction of the longitudinal coordinate and the numerical method of discrete orthogonalization along the radial coordinate. The boundary conditions at singular point $r=0$ of continuous-inhomogeneous solid cylinder are formulated. An analysis of the stress state of the solid cylinders depending on the variant of the change of elastic characteristics of the material and the different boundary conditions is carried out. It is shown that the greatest influence of the law of change of Young's modulus on the stress state of cylinders is observed for circumferential stresses on the outer surface in the average length section for both methods of the ends fixing. In addition, the influence of the material occurs for both circumferential and radial stresses on the ends for short cylinders ($l=6l_0$) with the rigidly fixed ends. The comparative analysis of the stress distribution for the different variants of the mechanical properties of the continuously inhomogeneous solid cylinder of the finite length is carried out. There are edge effects at the ends, which depend on the length of the cylinder with conditions of rigid fastening at the ends. The given results can be used in the strength calculations of the cylindrical elements of the modern structures.

Keywords: *axisymmetric problem, stress state, solid cylinders, continuous-heterogeneous materials, numerical method.*

Introduction

Solving problems of the theory of elasticity about the stress state of continuously inhomogeneous bodies requires the improvement of existing ones and the development of new effective methods that allow to fully take into account arbitrary dependences of material properties on coordinates and the nature of the applied load [1–5]. In papers [6–8], an approach to solving problems about the stress state of continuous radially inhomogeneous cylinders, which is based on the method of reduction to the integral Volterra equation, is proposed. To determine the stress-strain state of a finite cylinder under the action of compressive forces, the numerical-analytical method of finite squares is applied in paper [9]. Based on variational principles [10], problems about the stress state of an axisymmetric cylinder under the action of surface load [11] and an anisotropic thick-walled composite layered shell under the action of lateral pressure [12] were solved.

The need to assess the strength, durability and reliability of existing and newly created engineering systems becomes the reason for the emergence of complex problems of the mechanics of deformable solids. Their solution became possible due to the development of numerical methods in combination with the use of computer modeling of the posed problems [13–15]. One of the most frequently used numerical methods is the finite element method [16], for which the corresponding application software packages, focused on solving specific classes of problems of the theory of elasticity, have been developed and improved [17].

In the study of the stress state and vibrations of shells of various geometries and structures, the spline collocation method [18, 19], which allows obtaining solutions to the aforementioned classes of problems with a sufficient degree of accuracy, has come in handy. Thus, in [20], within the framework of the spatial theory of elasticity, the problem of the stress state of an inhomogeneous hollow cylinder with rigidly fixed ends was solved on the basis of spline approximation, while the reliability of the obtained results was checked using the finite element method. Based on the spline approximation method, in [21], the stress state of isotropic solid cylinders with different methods of ends fixing under the action of an external uniform normal load is studied.

This paper is a continuation of the papers devoted to the application of the spline approximation method to the solution of axisymmetric problems of the stress state of solid cylinders with different methods of the ends fixing. In this case, cylinders made of continuously inhomogeneous material are considered.

The **aim** of this paper is to study the effect of changes in the law of elastic properties of the material, the length of the cylinders, and the method of the ends fixing on the stress state of solid cylinders subjected to surface normal loading. The study is based on a methodology that utilizes analytical methods of variable separation with spline approximation of functions along the longitudinal coordinate and the numerical method of discrete orthogonalization along the radial coordinate [21].

Problem statement and solution methodology

An axisymmetric problem of linear elasticity theory is solved. Solid cylinders are referred to an orthogonal cylindrical coordinate system r, θ, z, where r is the polar radius, θ is the central angle in the cross section, and z is the longitudinal coordinate. The Lamé coefficients in this coordinate system take the form

$$H_1=1; \quad H_2=r; \quad H_3=1.$$

In this case, the following relations occur

$$x = r\cdot\cos\theta; \quad y = r\cdot\sin\theta; \quad z = z.$$

The equations of linear elasticity theory for an isotropic axisymmetric body in a cylindrical coordinate system [21] are taken as the starting points. Adding the load on the lateral surface $r=R$ and the boundary conditions at the ends $z=0$; l to them, we come at a two-dimensional boundary value problem.

For the cylinders under consideration, we choose a material that is continuously inhomogeneous in the radial direction. Let's assume the cylinders to be under the action of an external normal load $q=q_0\cdot\sin(\pi s/l)$ ($q_0=const$). The boundary conditions on the lateral surface $r=R$ due to the applied load have the form

$$\sigma_r = q_r; \ \tau_{rz} = 0 \ \text{at } r=R. \tag{1}$$

We will consider two types of boundary conditions at the ends, namely: hinged support and rigid fixation. In the case of hinged support of the ends, the boundary conditions have the form

$$\sigma_z = 0; \quad u_r = 0 \ \text{at } z=0;\ l, \tag{2}$$

and in the case of the rigidly fixed ends

$$u_r = 0; \quad u_z = 0 \ \text{at } z=0;\ l. \tag{3}$$

The problem is solved in the interval $0\le r\le R$, therefore it is also necessary to formulate boundary conditions for $r=0$. Based on physical considerations, these can be taken as

$$\tau_{rz} = 0; \quad u_r = 0 \ \text{at } r=0. \tag{4}$$

We choose the radial u_r and longitudinal u_z displacements as the solution functions. After some transformations from the original equations, we obtain a solution system of partial differential equations with variable coefficients along the radial coordinate r

$$\frac{\partial^2 u_r}{\partial r^2} = -\frac{a_3}{a_1}\frac{\partial^2 u_r}{\partial z^2} - \frac{a_2+a_3}{a_1}\frac{\partial^2 u_z}{\partial r\partial z} - \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{u}{r^2} - \frac{1}{a_1}\frac{\partial a_1}{\partial r}\frac{\partial u_r}{\partial r} - \frac{1}{a_1}\frac{\partial a_2}{\partial r}\frac{u_r}{r} - \frac{1}{a_1}\frac{\partial a_2}{\partial r}\frac{\partial u_z}{\partial z};$$

$$\frac{\partial^2 u_z}{\partial r^2} = -\frac{a_2+a_3}{a_3}\frac{\partial^2 u_r}{\partial r\partial z} - \frac{a_2+a_3}{a_3}\frac{1}{r}\frac{\partial u_r}{\partial z} - \frac{a_1}{a_3}\frac{\partial^2 u_z}{\partial z^2} - \frac{1}{r}\frac{\partial u_z}{\partial r} - \frac{1}{a_3}\frac{\partial a_3}{\partial r}\frac{\partial u_r}{\partial z} - \frac{1}{a_3}\frac{\partial a_3}{\partial r}\frac{\partial u_z}{\partial r}; \tag{5}$$

$$a_1 = \frac{E(1-\nu)}{1-\nu-2\nu^2}; \quad a_2 = \frac{E\nu}{1-\nu-2\nu^2}; \quad a_3 = \frac{E}{2(1+\nu)}; \ (0 \le r \le R;\ 0 \le z \le L),$$

where, in the general case, $E=E(r)$ is the Young's modulus; $\nu=\nu(r)$ is the Poisson's ratio.

Considering that

$$\sigma_r = a_1 \frac{\partial u_r}{\partial r} + a_2 \frac{u_r}{r} + a_2 \frac{\partial u_z}{\partial z}; \quad \sigma_\theta = a_2 \frac{\partial u_r}{\partial r} + a_1 \frac{u_r}{r} + a_2 \frac{\partial u_z}{\partial z}; \ \tau_{rz} = a_3\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),$$

boundary conditions (1), (4) in the displacements will have the form

$$u_r = 0; \quad \frac{\partial u_z}{\partial r} = 0 \ \text{at } r=0 \tag{6}$$

$$a_1 \frac{\partial u_r}{\partial r} + a_2 \frac{u_r}{r} + a_2 \frac{\partial u_z}{\partial z} = q_r; \quad \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = 0 \ \text{at } r=R.$$

To reduce the dimensionality of the boundary value problem for the system of partial differential equations (5) with boundary conditions (6), we give the solution of this system in the form of spline functions [21]

$$u_r = \sum_{i=1}^{N} u_{1i}(r)\cdot\varphi_{1i}(z); \quad u_z = \sum_{i=1}^{N} u_{2i}(r)\cdot\varphi_{2i}(z), \tag{7}$$

where $u_{1i}(r)$, $u_{2i}(r)$ are sought functions; $\varphi_{1i}(z)$, $\varphi_{2i}(z)$ are functions constructed using linear combinations of third-degree B-splines [22], which allow for an accurate satisfaction of the boundary conditions at the ends of the cylinder (2), or (3).

In the case of hinged fixing of the ends (2), the functions $\varphi_{ij}(z)$ (i=1,2; $j=\overline{0,N}$) are defined by expressions

$$\varphi_{10}(z) = -4B_3^{-1}(z) + B_3^0(z); \quad \varphi_{11}(z) = B_3^{-1}(z) - 0.5B_3^0(z) + B_3^1(z); \quad \varphi_{1j}(z) = B_3^j(z); \quad (j=2, 3, \ldots, N-2);$$

$$\varphi_{1N-1}(z) = B_3^{N-1}(z) - 0.5B_3^N(z) + B_3^{N+1}(z); \quad \varphi_{1N}(z) = -4B_3^N(z) + B_3^{N+1}(z);$$

$$\varphi_{20}(z) = B_3^0(z); \ \varphi_{21}(z) = B_3^{-1}(z) - 0.5B_3^0(z) + B_3^1(z); \quad \varphi_{2j}(z) = B_3^j(z); \quad (j=2, 3, \ldots, N-2); \tag{8}$$

$$\varphi_{2N-1}(z) = B_3^{N-1}(z) - 0.5B_3^N(z) + B_3^{N+1}(z); \quad \varphi_{2N}(z) = -4B_3^N(z) + B_3^{N+1}(z).$$

For the rigidly fixed ends (3), respectively,

$$\varphi_{i0}(z) = B_3^0(z); \quad \varphi_{i1}(z) = B_3^{-1}(z) - 0.5B_3^0(z) + B_3^1(z); \quad \varphi_{ij}(z) = B_3^j(z); \quad (j=2, 3, \ldots, N-2);\ (i=1, 2) \tag{9}$$

$$\varphi_{iN-1}(z) = B_3^{N-1}(z) - 0.5B_3^N(z) + B_3^{N+1}(z); \quad \varphi_{jN}(z) = -4B_3^N(z) + B_3^{N+1}(z).$$

After substituting expressions (7) taking into account (8), (9) into the system of differential equations (5), it is necessary to satisfy them at the collocation points $z=z_k$ ($k=\overline{0,N}$). In this case, we obtain the system 2(N+1) of ordinary differential equations. The same is done with the boundary conditions (7) on the surfaces r=0; R.

Let's assume that the collocation nodes ξ_k(k=0, 1, ..., N) satisfy the conditions

$$\xi_{2i} \in [z_{2i}, z_{2i+1}]; \quad \xi_{2i+1} \in [z_{2i}, z_{2i+1}] \quad (i=0, 1, 2,\ldots, n).$$

Then on each segment $[z_{2i}, z_{2i+1}]$ there will be two collocation nodes, and on adjacent segments $[z_{2i+1}, z_{2i+2}]$ there will be none at all. On each of the segments $[z_{2i}, z_{2i+1}]$ we will choose the collocation points as follows

$$\xi_{2i} = z_{2i} + t_1 h; \quad \xi_{2i+1} = z_{2i} + t_2 h \quad (i=0, 1, 2,\ldots, n),$$

where h is the uniform grid step on a segment [0, l]; t_1 and t_2 are roots of a second-order Legendre polynomial on a segment [0, 1]

$$t_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}; \quad t_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}.$$

Such collocation nodes are called optimal and allow obtaining an approximate solution to problem (6), (7) with an accuracy of $O(h^3)$.

Thus, the solution system of ordinary differential equations with coefficients varying along the coordinate r takes the form

$$\sum_{i=0}^{N}\frac{d^2u_{1i}}{dr^2}\cdot\varphi_{1i}(z_k)=-\frac{a_3}{a_1}\sum_{i=0}^{N}u_{1i}\cdot\varphi''_{1i}(z_k)-\frac{a_2+a_3}{a_1}\sum_{i=0}^{N}\frac{du_{2i}}{dr}\cdot\varphi'_{2i}(z_k)-\frac{1}{r}\sum_{i=0}^{N}\frac{du_{1i}}{dr}\cdot\varphi_{1i}(z_k)+$$

$$+\frac{1}{r^2}\sum_{i=0}^{N}u_{1i}\cdot\varphi_{1i}(z_k)-\frac{1}{a_1}\frac{da_1}{dr}\sum_{i=0}^{N}\frac{du_{1i}}{dr}\cdot\varphi_{1i}(z_k)-\frac{1}{a_1}\frac{da_2}{dr}\sum_{i=0}^{N}\frac{u_{1i}}{r}\cdot\varphi_{1i}(z_k)-\frac{1}{a_1}\frac{da_2}{dr}\sum_{i=0}^{N}u_{2i}\cdot\varphi'_{2i}(z_k);\quad(10)$$

$$\sum_{i=0}^{N}\frac{d^2u_{2i}}{dr^2}\cdot\varphi_{2i}(z_k)=-\frac{a_2+a_3}{a_3}\sum_{i=0}^{N}\frac{du_{1i}}{dr}\cdot\varphi'_{1i}(z_k)-\frac{a_2+a_3}{a_3}\frac{1}{r}\sum_{i=0}^{N}u_{1i}\cdot\varphi'_{1i}(z_k)-\frac{a_1}{a_3}\sum_{i=0}^{N}u_{2i}\cdot\varphi''_{2i}(z_k)-$$

$$-\frac{1}{r}\sum_{i=0}^{N}\frac{du_{2i}}{dr}\cdot\varphi_{2i}(z_k)-\frac{1}{a_3}\frac{da_3}{dr}\sum_{i=0}^{N}u_{1i}\cdot\varphi'_{1i}(z_k)-\frac{1}{a_3}\frac{da_3}{dr}\sum_{i=0}^{N}\frac{du_{2i}}{dr}\cdot\varphi_{2i}(z_k)\quad(k=2(N+1))$$

with boundary conditions

$$\sum_{i=0}^{N}u_{1i}\cdot\varphi_{1i}(z_k)=0;\quad \sum_{i=0}^{N}\frac{du_{2i}}{dr}\cdot\varphi_{2i}(z_k)=0\quad \text{at } r=0;$$

$$a_1\sum_{i=0}^{N}\frac{du_{1i}}{dr}\cdot\varphi_{1i}(z_k)+a_2\sum_{i=0}^{N}u_{1i}\cdot\varphi_{1i}(z_k)+a_2\sum_{i=0}^{N}u_{2i}\cdot\varphi'_{2i}(z_k)=q_r;$$

$$\sum_{i=0}^{N}u_{1i}\cdot\varphi'_{1i}(z_k)+\sum_{i=0}^{N}\frac{du_{2i}}{dr}\cdot\varphi_{2i}(z_k)=0\quad \text{at } r=R.\quad(11)$$

The resulting system of ordinary differential equations (10) with boundary conditions (11) forms a two-point boundary value problem in the interval $0 \le r \le R$. In this case, the system of equations (10) contains some terms that, when $r=0$, are transformed into uncertainty 0/0, to reveal which we will use the corresponding limit transitions at $r \to 0$, namely

$$\frac{u_{1i}}{r}\to\frac{du_{1i}}{dr}\quad (i=\overline{0,N}).\quad(12)$$

Taking into account (12), equations (10) at the point $r=0$ takes the form

$$\sum_{i=0}^{N}\frac{d^2u_{1i}}{dr^2}\cdot\varphi_{1i}(z_k)=-\frac{2}{a_1}\frac{da_1}{dr}\sum_{i=0}^{N}\frac{du_{1i}}{dr}\cdot\varphi_{1i}(z_k)-\frac{1}{a_1}\frac{da_2}{dr}\sum_{i=0}^{N}u_{2i}\cdot\varphi'_{2i}(z_k);$$

$$\sum_{i=0}^{N}\frac{d^2u_{2i}}{dr^2}\cdot\varphi_{2i}(z_k)=-\frac{a_1}{2a_3}\sum_{i=0}^{N}u_{2i}\cdot\varphi''_{2i}(z_k)-\frac{a_2+a_3}{a_3}.\quad(13)$$

By adding boundary conditions (11) to the systems of equations (10), (13), we arrive at a boundary value problem that can be solved numerically. At the same time, when $r=0$, the system of equations (13) is used, and for all other values of r – the system of equations (10) is used.

Let's introduce the notation

$$y_{1i}=u_{1i};\quad y_{2i}=\frac{du_{2i}}{dr};\quad y_{3i}=u_{2i};\quad y_{4i}=\frac{du_{1i}}{dr}\quad (i=\overline{0,N}).$$

Then the system of differential equations (10) to be solved can be given in vector form

$$\frac{d\overline{Y}}{dr}=A(r)\overline{Y}+\bar{f};\quad (0\le r\le R),\quad(14)$$

where $\overline{Y}=\{y_{10},...,y_{1N},y_{20},...,y_{2N},y_{30},...,y_{3N},y_{40},...,y_{4N}\}^{\mathrm{T}}$; $A(r)$ is the square matrix of order $4(N+1)\times 4(N+1)$; $\bar{f}$ is the vector of the right-hand side. The boundary conditions can be written similarly

$$B_1\overline{Y}(0)=\bar{b}_1;\quad B_2\overline{Y}(0)=\bar{b}_2,\quad(15)$$

where B_1, B_2 are rectangular matrices of order $2(N+1)\times 4(N+1)$.

To find the solution to the boundary value problem (14), (15), a stable numerical method of discrete orthogonalization is used.

Some estimates of the accuracy of the obtained results

Under the conditions of hinged fixing of the ends, this problem can be solved in another way. We choose two components of stresses and displacements as the functions to be solved [21]. Having separated the variables in the direction of the coordinate z, based on the representation of these functions in the form of expansions in Fourier series and, having solved the one-dimensional boundary value problem for the system of ordinary differential equations by the stable numerical method of discrete orthogonalization, we obtain a solution that we can accept as exact with a sufficient degree of accuracy.

For the cylinders under consideration, a polymeric continuous-inhomogeneous material with a gradient profile corresponding to the quadratic law of change of Young's modulus along the coordinate r: $E(r)=a\cdot r^2+b\cdot r+c$ ($0\le r\le R$) [23]. Due to the insignificant differences in Poisson's ratio for polymeric continuous-inhomogeneous materials, $\nu=0.4$ was chosen for its value. Let us have an increasing law of change of the elastic modulus, i.e. $E(0)=110$ MPa; $E(R/2)=150$ MPa; $E(R)=243$ MPa, then the coefficients $a=4.42$; $b=5.4$; $c=110$. The problem is solved for the following geometric parameters of the cylinders: radius $R=5\cdot l_0$, length $l=6\cdot l_0$; $10\cdot l_0$; $14\cdot l_0$.

In the following, all linear dimensions are referred to a unit of length, stresses – to a unit load.

The results of solving the problem for the maximum values of stresses σ_r, σ_θ are given in Table 1 in the average cross section of the cylinder length for three values of the coordinate $r=0.5\cdot R$; $0.75\cdot R$; R. The number I indicates the solutions obtained on the basis of the method of variables separation using Fourier series, the number II – on the basis of spline approximation for different numbers N_s of approximating spline functions. As can be seen from the given results, for $N_s>4$ the error does not exceed 1% for all values of the parameter N_s. Similar results are obtained for the other two laws of change of Young's modulus. In further calculations, $N_s=12$ is selected.

Table 1. Convergence of the solution depending on the number of spline functions

l	r	σ_r					σ_θ					Error, %	
		I	II				I	II					
			N_s					N_s				N_s	
			4	6	8	12		4	6	8	12	4	6–12
6	$0.5\cdot R$	6.61	3.22	6.60	6.59	6.61	6.92	3.36	6.91	6.90	6.92	>50	<1
	$0.75\cdot R$	8.29	4.09	8.28	8.28	8.29	9.48	4.64	9.46	9.46	9.48		
	R	10.00	5.01	9.99	10.00	10.00	14.39	7.12	14.33	14.38	14.39		
10	$0.5\cdot R$	8.49	4.19	8.49	8.48	8.49	9.21	4.53	9.19	9.19	9.20		
	$0.75\cdot R$	9.16	4.55	9.15	9.14	9.16	10.82	5.35	10.80	10.81	10.82		
	R	10.00	5.03	9.99	9.99	10.00	13.42	6.70	13.38	13.40	13.42		
14	$0.5\cdot R$	9.01	4.47	9.00	8.99	9.01	9.68	4.79	9.67	9.66	9.68		
	$0.75\cdot R$	9.43	4.71	9.41	9.41	9.43	10.86	5.40	10.85	10.84	10.86		
	R	10.00	5.04	9.99	9.98	10.00	12.59	6.32	12.57	12.57	12.59		

Numerical results and their analysis

Based on the given methodology, a study of the stress state of continuous inhomogeneous cylinders of different lengths along the thickness, which are under the action of an external load $q=q_0\cdot\sin(\pi s/l)$ ($q_0=10$) under the conditions of hinged and rigid fixing of the ends, was carried out. Three variants of the law of change of the elastic modulus are considered:

1) $E(0)=110$ MPa; $E(R/2)=150$ MPa; $E(R)=243$ MPa;

2) decreasing Young's modulus $E(0)=243$ MPa; $E(R/2)=150$ MPa; $E(R)=110$ MPa;

3) averaged over the thickness Young's modulus $E(0)=158.33$ MPa; Poisson's ratio $\nu=0.4$.

The problem was solved with the following initial data: the radius of the cylinder $R=5\cdot l_0$, its length $l=6\cdot l_0$; $10\cdot l_0$; $14\cdot l_0$, coefficients 1) $a=4.42$; $b=5.4$; $c=110$ – for the increasing; 2) $a=4.42$; $b=-47.8$; $c=110$ – for the decreasing; 3) $a=0$; $b=0$; $c=158.33$ – for the averaged laws of change of Young's modulus.

The results of solving the problem are given in Figs. 1–5 in the form of graphs of the distribution of the fields of circular σ_θ and radial σ_r stresses along the length of the cylinder for three sections along the radius: on the outer surface *(r=R)* – in Figs. 1–2; in the section $r=R/2$ – in Figs. 3–4 and in the section $r=0$ – in Fig. 5, which show the stress fields σ_θ and σ_r, where they have the same values.

The curves marked on the graphs with a solid line correspond to the Young's modulus averaged over the thickness, ones marked with the dashed line – to the increasing law of change of the elastic modulus, and ones marked with the dash-dotted line – to the decreasing one. Fig. 1, Fig. 3 show the distribution graphs of the circular stress fields, Fig. 2, Fig. 4 – for radial ones. In all figures, graphs *a*, *c*, *d* correspond to the case of rigid fixing of the ends, graphs *b*, *d*, *e* – for hinged support. In this case, options *a*, *b* correspond to the distribution of stresses for cylinders with a length of l=6, options *c*, *d* – for cylinder with the lengths of l=10, options *d*, *e* – for l=14.

The graphs shown in Figs. 1–5 illustrate the influence of the length of the cylinders, the method of the ends fixing and the characteristics of the material on the stress state of solid cylinders in different sections along the radius.

From Figs. 1–2 it is seen that the circular stresses are predominant in this section.

Their maximum amplitude values of the stresses σ_θ and σ_r acquire in the average section of length $z=l/2$ for both methods of the ends fixing, for all laws of change of the modulus of elasticity and all values of the cylinder length. As can be seen from the graphs (Figs. 1–2), the influence of the material on the stressed state of the cylinders under consideration takes place for circular stresses (Fig. 1) in the average length interval $l/6 \le z \le 5 \cdot l/6$ for all values of l for the two methods of the ends fixing and for the length l=6 at the ends of the cylinder.

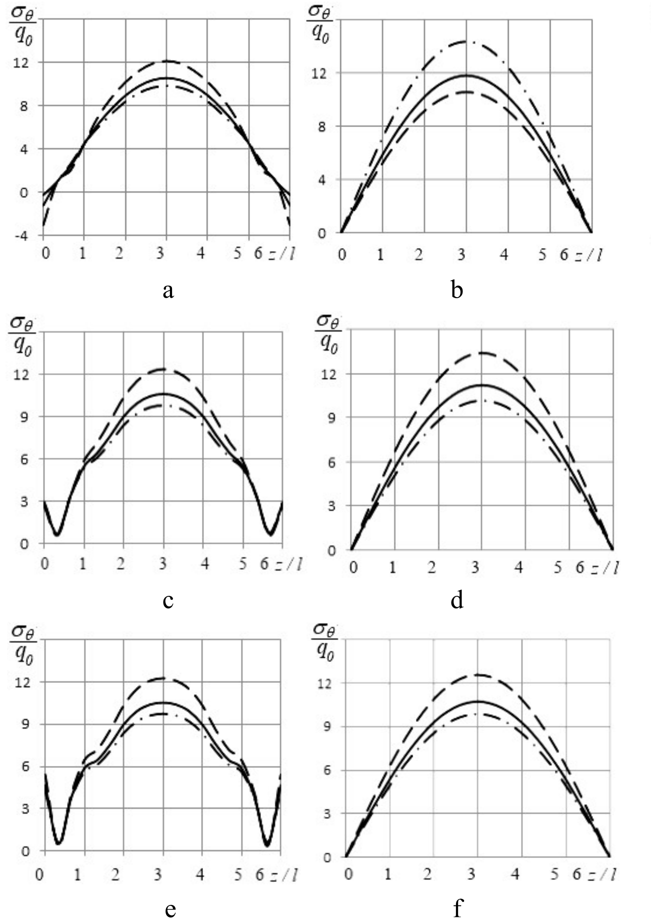

Fig. 1. Distribution of circumferential stresses along the length of the cylinders on the outer surface of the cylinder r=R

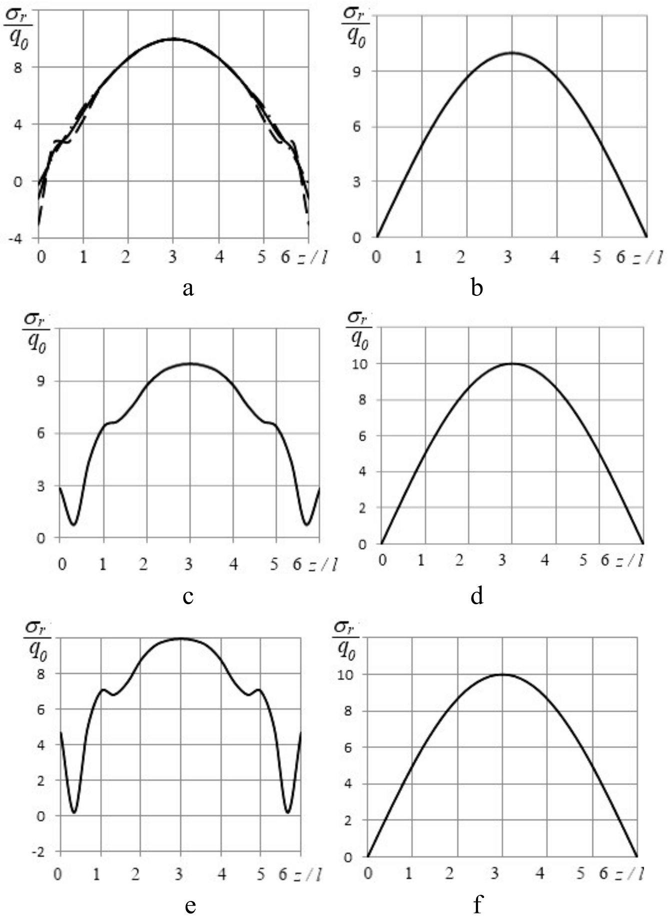

Figure 2. Distribution of radial stresses along the length of the cylinders on the outer surface of the cylinder r=R

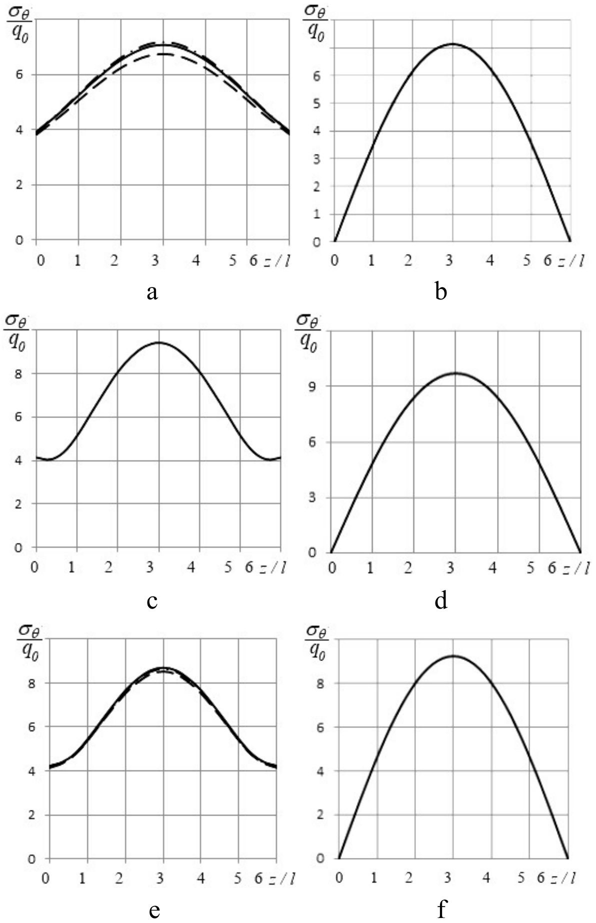

Fig. 3. Distribution of circular stresses along the length of the cylinders in cross section r=R/2

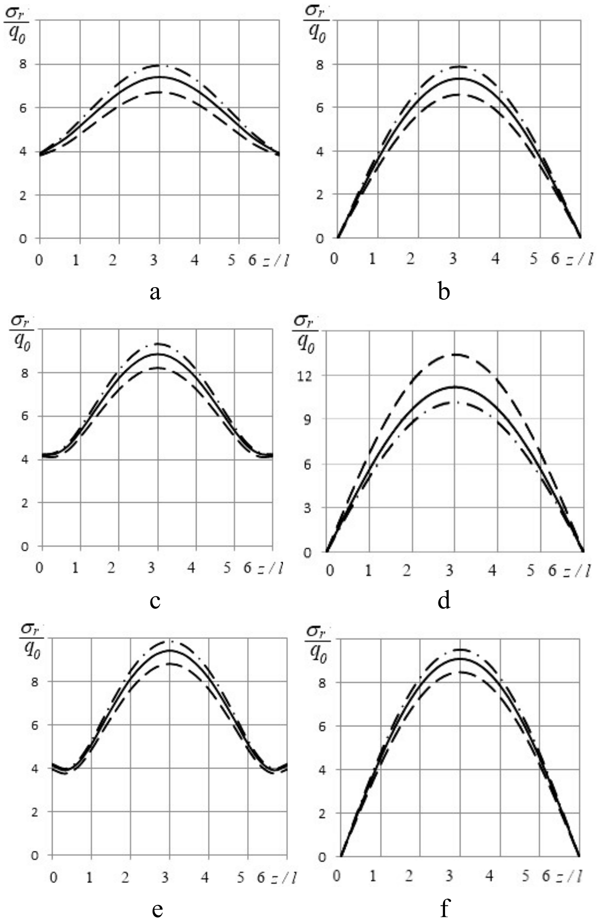

Fig. 4. Distribution of radial stresses along the length of the cylinders in cross section r=R/2

In the average length section, the maximum values of the circular stresses at the hinged fixing of the ends increase for the increasing law of change of the modulus of elasticity by 1.2 times compared to the Young's modulus averaged over the thickness and decreases by approximately 10% for the decreasing one.

In all variants of the law of change of the elastic modulus, the maximum value of the stresses is significantly affected by the increase in length, it decreases within 7–8%.

With the rigidly fixed ends, the value of the circumferential stresses does not depend on the length of the cylinders. In this case, the maximum value of the stresses compared to the average law of change of Young's modulus increases by 14% for the increasing law and decreases by approximately 8% for the decreasing one.

Under the conditions of hinged fixation in the average cross-section, the value of σ_θ increases by approximately 20% compared to rigid fixation.

The value of the circumferential stresses is significantly affected by the change in length in the case of rigid fixation near the ends of the cylinder. Thus, if for short cylinders l=6 the value of the stresses decreases rapidly monotonically, then for cylinders which length is l=10; 14 their value, approaching the ends, decreases almost to zero, and then increases rapidly at the ends.

For radial stresses (Fig. 2), the influence of the material takes place at the ends in the case of their rigid fixation for a short cylinder. Compared with the averaged law of change of Young's modulus, their amplitude value decreases by almost 5 times for the decreasing one and increases by approximately 3 times for the increasing law of change of the modulus of elasticity.

With the hinged method of the ends fixing, changes in length and the law of Young's modulus do not affect the distribution of radial stresses. Near the ends, when they are rigidly fixed, the length similarly affects the distribution of the fields of circular stresses.

When distributing the fields of circular stresses in the cross section $r=R/2$ (Fig. 3), the following features occur: for the hinged and rigid methods of the ends fixing, a change in the law of the modulus of elasticity does not affect their distribution. Increasing the length of the cylinders leads to an increase in the maximum stresses by 1.3 times for l=10 compared to the length l=6. For cylinders with a length of l=14, the magnitude of the stresses does not change significantly with hinged ends fixing, and in the case of rigid fixing - by approximately 8%.

The parabolic shape of the distribution curves of both circular and radial stresses, which occurs in the case of hinged ends fixing in all sections of the radial coordinate, is violated for rigidly fixed ends, and the monotonicity of the decline near the ends – with an increase in length.

In the average cross section of the radial coordinate, with an increase in the length of the cylinders, the influence of the material on the distribution of radial stresses is observed (Fig. 4).

In this case, the zone of influence decreases towards the middle of the length interval for both options for the ends fixing. In addition, in the case of the rigidly fixed ends, the difference in the values of the stresses along the length of the cylinder in the zone of the ends and the middle section becomes larger with increasing length for l=6 by approximately 2 times, for l=10 by 2.2 times and for l=14 by approximately 2.4 times.

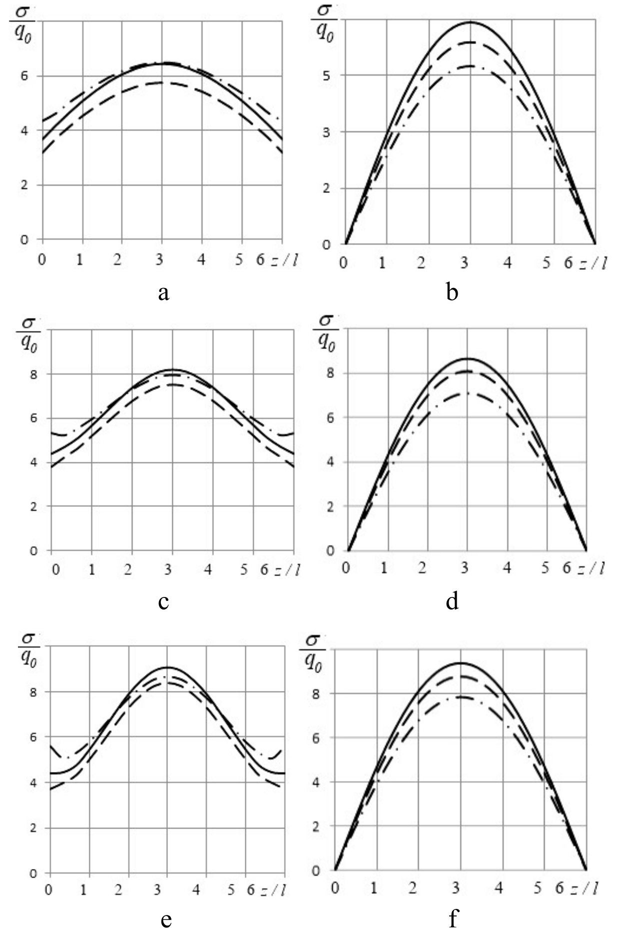

Fig. 5. Distribution of circumferential and radial stresses along the length of cylinders in cross section r=0

In the section r=0 (Fig. 5), where the circular and radial stresses have the same values, their magnitude is quantitatively and qualitatively affected by both the material, the length of the cylinder, and the method of the ends fixing.

If on the outer surface the stresses have maximum values for the increasing law of change of the elasticity modulus, then in the section r=0 – for the averaged one. Moreover, when the ends are hinged for short cylinders l=6, the maximum value of the stresses decreases by approximately 18% for the increasing one and by 1.2 times for the decreasing law of change of Young's modulus compared to the average.

In the case of the ends rigidly fixed on the ends of the cylinder, the maximum values of the stresses occur for the decreasing law of change of Young's modulus, and their value decreases by approximately 1.2 times compared to the average law of change of the modulus of elasticity and by 1.4 times compared to the increasing law of change of Young's modulus.

Within one law of change of the elasticity modulus, an increase in the length of the cylinder leads to an increase in the maximum value of the stresses by 1.3 times for the rigid one and by 1.5 times for the hinged one for l=10, by 1.4 times for the rigid one and by 1.6 times for the hinged one for l=14 compared to the corresponding values for l=6.

Conclusions

1. Within the framework of the linear theory of elasticity for an axisymmetric body, the problem of the stress state of solid cylinders made of continuously inhomogeneous material, which are under the action of a uniform normal load with different methods of the ends fixing, is solved. In this case, an approach that is based on the use of the method of separation of variables using spline approximation of functions in the di-

rection of the longitudinal coordinate and numerical solution of the obtained one-dimensional boundary value problem by a stable numerical method of discrete orthogonalization is used.

2. Thanks to the use of appropriate limit transitions, the uncertainty (0/0) of some components of the system of ordinary differential equations being solved at a geometrically singular point of the cylinder (r=0) is revealed.

3. An analysis of the characteristics of the stress state of the cylinders under consideration for the distribution fields of circular and radial stresses depending on the law of change of the elasticity modulus, the length of the cylinders and the method of the ends fixing was carried out.

4. It was found that the greatest influence of the law of change of Young's modulus on the stressed state of cylinders occurs for circumferential stresses on the outer surface in the average length section for both methods of the ends fixing. In addition, the influence of the material is observed for circumferential and radial stresses on the ends for short cylinders ($l=6\cdot l_0$) with a rigid method of the ends fixing. Compared with the average law, their value decreases by about 5 times for a decreasing one and increases by about 3 times for an increasing law of change of the elasticity modulus.

5. Under conditions of the rigidly fixed ends, edge effects, which depend on the length of the cylinders, occur at the ends.

The results obtained in the paper can be used in calculations for the strength and reliability of structural elements and machine parts of a similar type.

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Дослідження напруженого стану суцільних циліндрів неоднорідної структури за різних граничних умов на торцях

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Розв'язання задач теорії пружності про напружений стан неперервно-неоднорідних тіл потребує удосконалення існуючих і розробки нових чисельно-аналітичних методів, що дають змогу повною мірою врахувати довільні залежності властивостей матеріалу від координат і характер прикладеного навантаження. Стаття присвячена розв'язанню вісесиметричної задачі лінійної теорії пружності про рівновагу суцільних, неоднорідних вздовж радіальної координати циліндрів, за різних способів закріплення торців. Як матеріал обрано полімерний неперервно-неоднорідний із градієнтним профілем, що відповідає квадратичному закону зміни модуля Юнга вздовж радіальної координати. Розглянуто три варіанти закону зміни модуля пружності (зростаючий, спадний та усереднений) і два способи закріплення торців (шарнірне обпирання і жорстке закріплення). Метою публікації є проведення чисельного аналізу напруженого стану циліндрів даного класу залежно від закону зміни пружних *властивостей матеріалу, довжини циліндрів і способу закріплення торців. Розв'язок задачі базується на застосуванні методу сплайн-апроксимації функцій в напрямку поздовжньої координати й чисельного методу дискретної ортогоналізації за радіальною координатою. Розкрито невизначеність у геометрично особливій точці r=0. Проаналізовано напружений стан циліндрів, що вивчаються, залежно від закону зміни пружних характеристик матеріалу, довжини циліндрів і способу закріплення торців. Показано, що найбільший вплив закону зміни модуля Юнга на напружений стан циліндрів спостерігається для колових напружень на зовнішній поверхні в середньому перерізі довжини для обох способів закріплення торців. Крім того, вплив матеріалу має місце як для колових, так і для радіальних напружень на торцях для коротких циліндрів ($l=6l_0$) за жорсткого способу закріплення торців. Порівняно з усередненим законом, їх величина зменшується приблизно у 5 разів для спадного і збільшується приблизно у 3 рази для зростаючого закону зміни модуля пружності. За умов жорсткого закріплення торців мають місце крайові ефекти на торцях, які залежать від довжини циліндрів. Отримані в роботі результати можуть бути використані при розрахунках на міцність елементів конструкцій та деталей машин подібного типу.*

***Ключові слова**: вісесиметрична задача, напружений стан, суцільні циліндри, неперервно-неоднорідні матеріали, чисельний метод.*

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