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ELASTICITY PROBLEM FOR A LAYER WITH A CYLINDRICAL CAVITY UNDER PERIODIC LOADING

¹ Tetiana M. Aloshechkinaatn4042@gmail.com

ORCID: 0000-0001-7234-1558

² Natalia A. Ukrainetsn.ukrainets@khai.edu

ORCID: 0000-0001-7406-5809

² Vitalii Yu. Miroshnikovv.miroshnikov@khai.edu

ORCID: 0000-0002-9491-0181

¹ O. M. Beketov National University
of Urban Economy in Kharkiv,
17, Chornohlazivska str., Kharkiv,
61002, Ukraine

² National Aerospace University
"Kharkiv Aviation Institute"
17, Vadyma Manka str., Kharkiv,
61070, Ukraine

In aerospace and mechanical engineering, elements that are loaded by periodic loads (periodic function) are used. In problems for a layer with cylindrical inhomogeneities, it is difficult to take such loads into account. Therefore, there is a need to develop a methodology for calculating the stress state for a layer with a cylindrical cavity and taking into account the boundary conditions in the form of a periodic function. In this paper, we propose a solution to the problem of elasticity theory for a layer with a cylindrical cavity within the framework of the generalized Fourier method. Stresses are given at the upper boundary of the layer and on the surface of the cylindrical cavity, and displacements are given at the lower boundary of the layer. The layer and cylindrical cavity are considered in different coordinate systems (Cartesian and cylindrical). The redistribution functions of the generalized Fourier method are applied to the Lamé equations. The problem is reduced to the sum of two solutions – an auxiliary problem and the main problem. Both problems are reduced to infinite systems of linear algebraic equations, which allow the application of the reduction method to them. After finding the unknowns in the auxiliary problem, the stresses at the geometric location of the cavity are found. The main problem is solved for the layer with the cavity, on which stresses obtained from the auxiliary problem are set with the reverse sign. The complete solution consists of the auxiliary and main problems. Having calculated all the unknowns, it is possible to obtain the stress-strain state at any point of the body with a given accuracy. Numerical analysis of the stress state showed high accuracy of the boundary conditions and dependence on periodic loading. Thus, the stresses σ_x and σ_z at the upper boundary of the layer have extremes in the places of maximum values σ_y and their negative values increase at the location of the cavity. At the same time, the stresses σ_x exceed the specified σ_y .

Keywords: periodic loading, layer with a cylindrical cavity, Lamé equation, generalized Fourier method.

Introduction

When designing various types of technological equipment in mechanical engineering, structures in aircraft, and during construction, it is often necessary to take into account periodic loading. However, this is associated with a number of difficulties. This statement is explained by the fact that one of the main features of this type of loading is its infinite nature, which cannot be taken into account by software packages based on the finite element method [1, 2].

Another problem is the calculation of elements with cylindrical inhomogeneities. Much attention has been paid to this area of research. Nowadays, there are a number of classical papers devoted to the analytical and analytical-numerical solution of problems for a half-space and a layer with a cylindrical cavity [3–5]. However, the proposed classical methods, based only on the Fourier series expansion, consider the problem either in a flat form or with fewer than three boundary conditions.

A number of papers consider problems for a layer with inhomogeneities perpendicular to it [6–11]. Thus, in papers [6] and [7], scientists studied the layer, simplifying the contact conditions on its upper and lower surfaces. However, this approach does not allow to study a layer with a cylindrical inclusion or cavity, since the use of integral Laplace and Fourier transforms leads to the formulation of a mathematical problem, the solution of which is exclusively the problem of wave diffraction and does not take into account the geometry of the inclusion/cavity.

The use of the genetic algorithm (GA), the gravity search algorithm (GSA) and the Bat algorithm (BA)

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for the analysis of the stresses of a perforated plate, as done in paper [8], provides only an approximate solution.

The problem of torsion of an elastic half-space containing a vertical cylindrical cavity and loaded by a coaxial stamp is considered in paper [9]. To solve this problem, the authors developed two original methods that allow reducing it to ordinary integral equations of the second kind. Despite the novelty of the approach, the obtained solution is approximate and shows some deviations from the solution known as the Reissner-Sagoci problem.

In [10], an analytical method for calculating composite laminated perforated plates based on the layer-beating method was proposed. The effectiveness and accuracy of this method were confirmed by comparing the obtained results with the results of finite element analysis. Another paper [11] is devoted to the study of torsional vibrations of a flat round stamp in contact with a multilayer elastic base containing a vertical cylindrical cavity. To solve this problem, the Weber integral transformation and the method of paired integral equations were used. It is important to note that the mentioned methods [10, 11] cannot be applied to solving problems with inhomogeneities located parallel to the layers.

For problems with a longitudinal cylindrical cavity, it is proposed to use the generalized Fourier method [12]. Its effectiveness was proven in papers [13–15] for a cylinder with cylindrical cavities or inclusions, in which the solution is presented as a superposition of the basic solutions of the Lamé equation for cylindrical geometry, where each coordinate system is connected with the center of the corresponding boundary surface of the body.

A mathematical justification of the formulas for the transition between the Cartesian and cylindrical coordinate systems in the problem for a half-space with a cylindrical cavity is given in paper [16].

The problem for a layer with a cylindrical cavity in displacements was solved in paper [17], with the cavity in stresses – in [18], and of mixed type – in [19]. However, the methods proposed in papers [17–19] can only take into account rapidly decreasing functions at the boundaries and are not able to take into account infinite ones, such as periodic functions. Consideration of periodic displacements in the second basic problem (in displacements) for a layer with a cylindrical cavity was proposed in [20].

This paper is aiming to:

- create a method for solving a mixed problem of the theory of elasticity, when a periodic stress function is given on the upper boundary of the layer, displacements – on the lower boundary of the layer, and stresses – on the cavity;
- analyze the stress state of the layer.

Problem statement

The model is a layer with a cylindrical cavity (Fig. 1).

The layer is considered in the Cartesian coordinate system (x, y, z) , the cavity is considered in the cylindrical coordinate system (ρ, φ, z) , which is equally oriented and connected to the coordinate system of the layer. The distance from the center of coordinates to the upper boundary of the layer is h , and to the lower boundary it is $\tilde{h}$. The radius of the cavity is R .

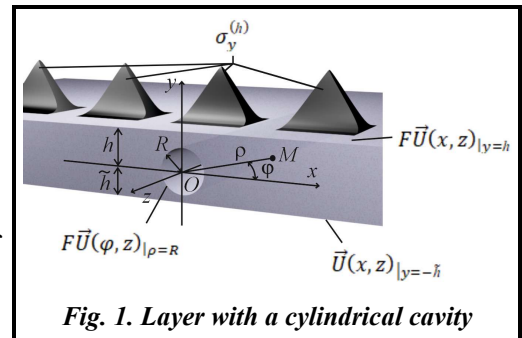


Fig. 1. Layer with a cylindrical cavity

The stresses $F\tilde{U}(x, z)|_{y=h} = \tilde{F}_h^0(x, z)$ are given at the upper boundary of the layer, displacements $\tilde{U}(x, z)|_{y=-\tilde{h}} = \tilde{F}_h^0(x, z)$ – on the bottom one, stresses $F\tilde{U}(\varphi, z)|_{\rho=R} = \tilde{F}_0^{(p)}(\varphi, z)$ – on the cylindrical cavity, where

$$\begin{aligned}\tilde{F}_h^0(x, z) &= \tau_{yx}^{(h)} \cdot \vec{e}_x + \sigma_y^{(h)} \cdot \vec{e}_y + \tau_{yz}^{(h)} \cdot \vec{e}_z; \\ \tilde{F}_h^0(x, z) &= U_x^{(\tilde{h})} \cdot \vec{e}_x + U_y^{(\tilde{h})} \cdot \vec{e}_y + U_z^{(\tilde{h})} \cdot \vec{e}_z; \\ \tilde{F}_0^{(p)}(\varphi, z) &= \sigma_\rho \cdot \vec{e}_\rho + \tau_{\rho\varphi} \cdot \vec{e}_\varphi + \tau_{\rho z} \cdot \vec{e}_z\end{aligned}$$

– known functions.

Functions $\tau_{yx}^{(h)}$, $\sigma_y^{(h)}$, $\tau_{yz}^{(h)}$ given on the upper boundary of the layer are periodic along the x -axis and rapidly decreasing along the z -axis. Other given functions are rapidly decreasing to zero along the z -axis for the cavity, and along the x - and z -axis for the lower boundary of the layer.

Solution method

Solution of the Lamé equation $\Delta \vec{u} + (1 - 2\sigma)^{-1} \nabla \operatorname{div} \vec{u} = 0$ is given as the sum of solutions of two problems $\vec{U} = \vec{U}_0 + \vec{U}_1$.

First problem ($\vec{U}_0$) – auxiliary, it is introduced to take into account the given function of periodic stresses. In this case, a layer without a cylindrical cavity and with periodic stresses given at the upper boundary of the layer, and with zero displacements – at the lower boundary of the layer, is considered. The solution has the following form:

$$\vec{U}_0 = \sum_{k=1}^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (H_{k,n}^{(0)}(\lambda) \cdot \vec{u}_k^{(+)}(x, y, z; \lambda, \mu_n) + \tilde{H}_{k,n}^{(0)}(\lambda) \cdot \vec{u}_k^{(-)}(x, y, z; \lambda, \mu_n)) d\lambda, \quad (1)$$

where $H_{k,n}^{(0)}(\lambda)$, $\tilde{H}_{k,n}^{(0)}(\lambda)$ are unknowns that must be found from the boundary conditions of this auxiliary problem; $\vec{u}_k^{(+)}(x, y, z; \lambda, \mu)$ i $\vec{u}_k^{(-)}(x, y, z; \lambda, \mu)$ are basic solutions of the Lamé equation for the layer [12]:

$$\vec{u}_k^{\pm}(x, y, z; \lambda, \mu) = N_k^{(d)} e^{i(\lambda z + \mu x) \pm \gamma y}; \quad k=1, 2, 3; \quad (2)$$

$N_1^{(d)} = \frac{1}{\lambda} \nabla$; $N_2^{(d)} = \frac{4}{\lambda} (v-1) \vec{e}_y^{(1)} + \frac{1}{\lambda} \nabla(y \cdot)$; $N_3^{(d)} = \frac{i}{\lambda} \operatorname{rot}(\vec{e}_z^{(1)} \cdot)$; $\gamma = \sqrt{\lambda^2 + \mu^2}$; $-\infty < \lambda, \mu < \infty$; $\vec{e}_x, \vec{e}_y, \vec{e}_z$ – are Cartesian coordinate system orts; v is the Poisson's ratio.

To create a system of equations, we substitute the known functions $\tau_{yx}^{(h)}$, $\sigma_y^{(h)}$, $\tau_{yz}^{(h)}$ into the left-hand side of (1), having previously represented them through a Fourier series along the x -axis, and a Fourier integral along the z -axis

$$\vec{F}_h^0(x, z) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \vec{c}_h(\lambda, n) \cdot e^{i(\mu_n x + \lambda z)} d\lambda,$$

where

$$\vec{c}_h(\lambda, n) = \frac{1}{4\pi\ell} \int_{-\ell}^{\ell} dx \int_{-\infty}^{\infty} \vec{F}_h^0(x, z) \cdot e^{-i(\mu_n x + \lambda z)} dz; \quad (3)$$

2ℓ is the period of function; $\mu_n = n\pi / \ell$.

Having freed the right and left parts from series, integrals and $e^{i(\mu_n x + \lambda z)}$, we obtain a system of linear algebraic equations for finding $H_{k,n}^{(0)}(\lambda)$, $\tilde{H}_{k,n}^{(0)}(\lambda)$

$$\begin{cases} \sum_{s=1}^3 H_{k,n}^{(0)}(\lambda) \cdot \vec{d}_s^{+}(h; \lambda, \mu_n) + \sum_{s=1}^3 \tilde{H}_{k,n}^{(0)}(\lambda) \cdot \vec{d}_s^{-}(h; \lambda, \mu_n) = \vec{c}_h(\lambda, n) \\ \sum_{s=1}^3 H_{k,n}^{(0)}(\lambda) \cdot \vec{d}_s^{+}(-\tilde{h}; \lambda, \mu_n) + \sum_{s=1}^3 \tilde{H}_{k,n}^{(0)}(\lambda) \cdot \vec{d}_s^{-}(-\tilde{h}; \lambda, \mu_n) = 0 \end{cases}, \quad (4)$$

where $\vec{d}_s^{\pm}(y; \lambda, \mu_n) = N_k^{(d)} \cdot e^{\pm \gamma y}$.

Equation (4) is projected onto the coordinate axes (the projections are equated with respect to the basis vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$), resulting in expressions for $H_{k,n}^{(0)}(\lambda)$ and $\tilde{H}_{k,n}^{(0)}(\lambda)$

$$H_{j,n}^{(0)}(\lambda) = \sum_{k=1}^3 \frac{A_{k,j}}{D} (\vec{c}_h(\lambda, n)) \cdot \vec{e}_k; \quad \tilde{H}_{j,n}^{(0)}(\lambda) = \sum_{k=1}^3 \frac{A_{k,j+3}}{D} (\vec{c}_h(\lambda, n)) \cdot \vec{e}_k,$$

where $j = 1, 2, 3$; $A_{1...6, 1...6}$ is the algebraic complement of a system of equations; D – determinant of the system of equations.

After determining the unknowns $H_{k,n}^{(0)}(\lambda)$ and $\tilde{H}_{k,n}^{(0)}(\lambda)$, the stresses at the geometric location of the cavity are found. To do this, the transition formulas from the basic solutions of the layer ($\vec{u}_k^{(+)}$ and $\vec{u}_k^{(-)}$) are applied to the internal basic solutions of the cylinder ($\vec{R}_{k,m}$) [12]

$$\bar{u}_k^{(\pm)}(x, y, z) = e^{i\mu\bar{x}_p \pm \gamma\bar{y}_p} \cdot \sum_{m=-\infty}^{\infty} (i \cdot \omega_{\mp})^m \bar{R}_{k,m}, \quad (k=1, 3);$$

$$\bar{u}_2^{(\pm)}(x, y, z) = e^{i\mu\bar{x}_p \pm \gamma\bar{y}_p} \cdot \sum_{m=-\infty}^{\infty} \left[(i \cdot \omega_{\mp})^m \cdot \lambda^{-2} \left((m \cdot \mu + \bar{y}_p \cdot \lambda^2) \cdot \bar{R}_{1,m} \pm \gamma \cdot \bar{R}_{2,m} + 4\mu(1-\sigma)\bar{R}_{3,m} \right) \right],$$

where $\bar{R}_{k,m} = \bar{b}_{k,m}(\rho_p, \lambda) \cdot e^{i(m\varphi_p + \lambda z)}$; $\bar{b}_{1,n}(\rho, \lambda) = \bar{e}_\rho \cdot I'_n(\lambda\rho) + i \cdot I_n(\lambda\rho) \cdot \left(\bar{e}_\varphi \frac{n}{\lambda\rho} + \bar{e}_z \right)$;

$$\bar{b}_{2,n}(\rho, \lambda) = \bar{e}_\rho \cdot [(4\sigma - 3) \cdot I'_n(\lambda\rho) + \lambda\rho I''_n(\lambda\rho)] + \bar{e}_\varphi i \cdot m \left(I'_n(\lambda\rho) + \frac{4(\sigma - 1)}{\lambda\rho} I_n(\lambda\rho) \right) + \bar{e}_z i\lambda\rho I'_n(\lambda\rho);$$

$$\bar{b}_{3,n}(\rho, \lambda) = - \left[\bar{e}_\rho \cdot I_n(\lambda\rho) \frac{n}{\lambda\rho} + \bar{e}_\varphi \cdot i \cdot I'_n(\lambda\rho) \right]; \quad \bar{e}_\rho, \bar{e}_\varphi, \bar{e}_z \text{ are cylindrical coordinate system orts.}$$

Having got rid of the integral over λ and $e^{i(\lambda z + m\varphi)}$, stress at the geometric location of the cavity was obtained

$$\bar{h}_m^0(\rho, \lambda) = \sum_{n=-\infty}^{\infty} \left[\sum_{s=1}^3 \left(r_{n,j}(R; m, \lambda) \cdot f_{k,n}^m(\lambda, \mu_n) \cdot \sum_{k=1}^3 \frac{A_{k,s}}{D} (\bar{c}_h(\lambda, n)) \right) + \sum_{s=1}^3 \left(r_{n,j}(R; m, \lambda) \cdot \tilde{f}_{k,n}^m(\lambda, \mu_n) \cdot \sum_{k=1}^3 \frac{A_{k,s+3}}{D} (\bar{c}_h(\lambda, n)) \right) \right],$$

$$\text{where } f_{k,n}^m(\lambda, \mu) = (i \cdot \omega_-(\lambda, \mu))^m \cdot \begin{pmatrix} 1 & 0 & 0 \\ \frac{m\mu}{\lambda^2} & \frac{\gamma}{\lambda^2} & \frac{4\mu(1-\sigma)}{\lambda^2} \\ 0 & 0 & 1 \end{pmatrix}; \quad \tilde{f}_{k,n}^m(\lambda, \mu) = (i \cdot \omega_+(\lambda, \mu))^m \cdot \begin{pmatrix} 1 & 0 & 0 \\ \frac{m\mu}{\lambda^2} & -\frac{\gamma}{\lambda^2} & \frac{4\mu(1-\sigma)}{\lambda^2} \\ 0 & 0 & 1 \end{pmatrix};$$

$r_{n,j}(R; m, \lambda)$ is the tensor obtained by applying the stress operator to $\bar{R}_{k,m}$; $\bar{c}_h(\lambda, n)$ – given in formula (3).

The main problem ($\bar{U}_1$) considers a layer with a cylindrical cavity with stresses given on the surface of the cavity with opposite signs $\bar{h}_m^{(0)}(\rho, \lambda)$. Solution for $\bar{U}_1$ looks like this [19]:

$$\bar{U}_1 = \sum_{k=1}^3 \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_{k,m}(\lambda) \cdot \bar{S}_{k,m}(\rho, \varphi, z; \lambda) d\lambda + \sum_{k=1}^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(H_k(\lambda, \mu) \cdot \bar{u}_k^{(+)}(y, z; \lambda, \mu) + \tilde{H}_k(\lambda, \mu) \cdot \bar{u}_k^{(-)}(y, z; \lambda, \mu) \right) \cdot d\mu \cdot d\lambda,$$

where $H_k(\lambda, \mu)$, $\tilde{H}_k(\lambda, \mu)$, $B_{k,m}(\lambda)$ are unknowns that need to be found; $\bar{u}_k^{(+)}(y, z; \lambda, \mu)$ and $\bar{u}_k^{(-)}(y, z; \lambda, \mu)$ are basic solutions (3); $\bar{S}_{k,m}(\rho, \varphi, z; \lambda)$ are basic solutions from outside the cylinder [12]

$$\bar{S}_{k,m}(\rho, \varphi, z; \lambda) = N_k^{(p)} \left[(\text{sign } \lambda)^m K_m(|\lambda| \cdot \rho) \cdot e^{i(\lambda z + m\varphi)} \right], \quad k=1, 2, 3;$$

$$N_1^{(p)} = \frac{1}{\lambda} \nabla; \quad N_2^{(p)} = \frac{1}{\lambda} \left[\nabla \left(\rho \frac{\partial}{\partial \rho} \right) + 4(\nu - 1) \left(\nabla - \bar{e}_z^{(2)} \frac{\partial}{\partial z} \right) \right]; \quad N_3^{(p)} = \frac{i}{\lambda} \text{rot}(\bar{e}_z^{(2)} \cdot); \quad -\infty < \lambda < \infty.$$

The method for solving this basic problem is the same as for [12].

After finding all the unknowns for $\bar{U}_0$ and $\bar{U}_1$, it is possible to obtain the stress-strain state at any point in the layer by applying the formulas for the transition of basic solutions between the Cartesian and cylindrical coordinate systems.

Numerical studies of the stressed state

The elastic isotropic layer has a cylindrical cavity with radius $R=10$ mm (Fig. 1). Physical characteristics of the layer: aluminum plate D16T, Poisson's ratio $\nu_0=0.3$, modulus of elasticity $E_0=7.1 \times 10^4$ MPa. Geometric parameters of the model: $h=\tilde{h}=16$ mm.

At the upper boundary of the layer along the x -axis, a periodic stress function is given in the form of triangles along the x -axis and in the form of waves along the z -axis

$$\sigma_y^{(h)}(x, z) = \begin{cases} -\left(1 - \frac{x}{2}\right) \cdot (10^4 \cdot (z^2 + 10^2)^{-2}), & \text{at } 0 \leq |x| \leq 2 \\ 0, & \text{at } 2 < |x| < \pi \end{cases}$$

The other parameters that are set include: tangential stresses $\tau_{yx}^{(h)} = \tau_{yz}^{(h)} = 0$, displacements $U_x^{(h)} = U_y^{(h)} = U_z^{(h)} = 0$ on the lower boundary, stresses $\sigma_p^{(p)} = \tau_{\rho\phi}^{(p)} = \tau_{\rho z}^{(p)} = 0$ on the surface of the cavity. Function $\sigma_y^{(h)}(x, z)$ image through the Fourier series along the x -axis and the Fourier integral along the z -axis has the form

$$f(x; \lambda, n) = -\left[\frac{2}{\pi} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\sin n}{n} \right)^2 \cos nx \right) \right] \cdot [2.5e^{-|\lambda|10} (|\lambda|10 + 1)].$$

The infinite system of equations was reduced to a finite one along the parameter $m=6$ and $n=35$. The accuracy of fulfilling the boundary conditions for the specified values of the geometric parameters is 10^{-4} for values that are equal to zero.

Stresses σ_x and σ_z on the upper boundary of the layer, which arise under the action of a periodic function σ_y given on this surface are shown in Fig. 2.

Periodic loading at the upper boundary of the cylindrical cavity leads to the development of compressive stresses σ_x and σ_z (Fig. 2). The stresses σ_x have a maximum value $\sigma_{x, \max} = -1.1689$ MPa at $x=0$, which exceeds the given $\sigma_y=1$ MPa. This is due to the concentration of three triangular loads from the periodic function in the cylindrical cavity area.

From Fig. 2 it can be seen that the negative values of the stresses σ_x and σ_z in the cavity area increase significantly.

The stresses σ_x and σ_z have extrema at the places of the maximum values σ_y (Fig. 2). At $\sigma_y=0$, the stresses σ_x acquire positive values.

The stresses σ_x and σ_z at $x=0$ on the upper boundary of the layer along the z -axis are shown in Fig. 3.

At $z=0$, the stresses σ_x reach their maximum value (Fig. 3), which coincides with Fig. 2. Regardless of z , these stresses are always compressive.

The stresses σ_z , in addition to negative values in the maximum load zone, have positive values in the damping zone σ_y . The difference between the functions along the x -axis and along the z -axis also affects the difference between the maximum values of the stresses σ_x and σ_z (Fig. 3).

The stresses σ_ϕ and σ_z on the surface of the cylindrical cavity at $z=0$ are shown in Fig. 4.

On the surface of the cylindrical cavity, the stresses σ_ϕ reach their maximum values at angles $\phi=0.585$ and $\phi=2.55$, $\sigma_{\phi \max}=0.49727$ MPa (Fig. 4). At $\phi=\pi/2$, the stresses σ_ϕ have positive but minimal values.

The stresses σ_z on the surface of the cylindrical cavity reach their maximum values at $\phi=\pi/2$ (Fig. 4).

The stresses σ_ρ , σ_ϕ and σ_z at the neck between the cylindrical cavity and the upper boundary of the layer are shown in Fig. 5.

At $y=10$ the stresses $\sigma_\rho=0.00003$, at $y=16$ the stresses $\sigma_\rho=0.99995$ (Fig. 5), which corresponds to the given and coincides with the stresses σ_y with an accuracy of 10^{-4} .

The stresses σ_ϕ and σ_z have extreme values at $y=10$ and $y=16$ (Fig. 5), which coincides with the values in Fig. 3 and Fig. 4.

The distribution of stresses σ_ρ , σ_ϕ and σ_z along the neck has a nonlinear character.

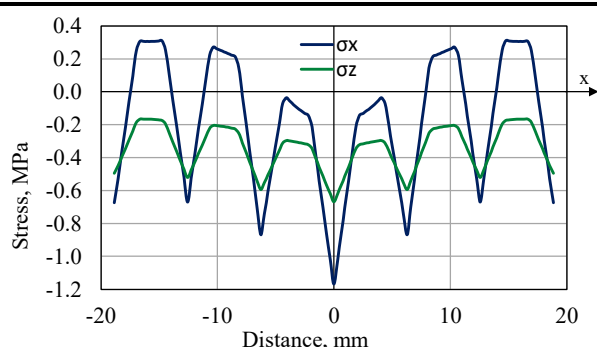


Fig. 2. Stresses σ_x and σ_z on the upper boundary of the layer along the x-axis

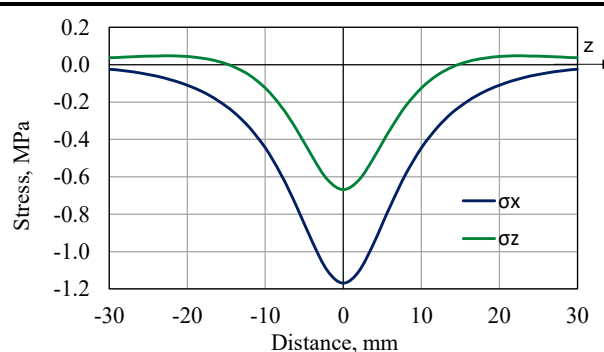


Fig. 3. Stresses σ_x and σ_z on the upper boundary of the layer along the z-axis

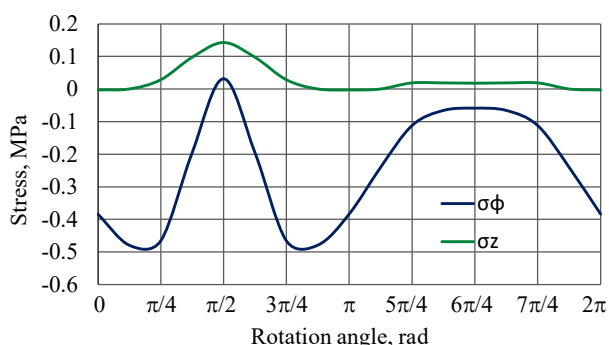


Fig. 4. Stresses σ_ϕ and σ_z on the surface of the cylindrical cavity

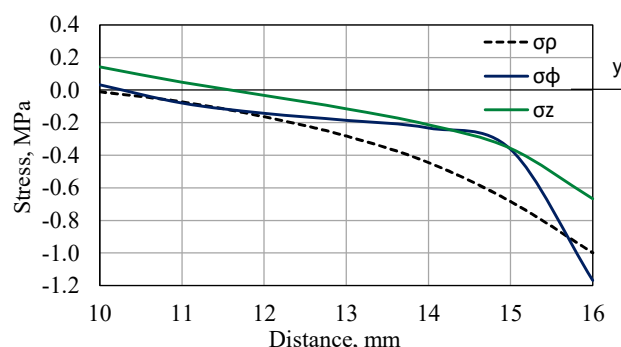


Fig. 5. Stresses σ_ρ , σ_ϕ and σ_z from the cavity surface to the upper boundary of the layer along the axis y

Conclusions

A method of consideration of periodic loads in a mixed problem of the theory of elasticity for a layer with a cylindrical cavity within the framework of the generalized Fourier method is proposed. The given auxiliary problem makes it possible to expand the given function into a Fourier series instead of integrating it.

The stress state for a layer with a cylindrical cavity is found for the given stresses (periodic function) on the upper boundary of the layer, displacements on the lower boundary, and stresses on the surface of the cavity.

The analysis of the stress state showed its dependence on the periodic loading. Thus, the stresses σ_x and σ_z on the upper boundary of the layer have extrema at the locations of the maximum values σ_y , and their negative values increase at the location of the cavity. The stresses σ_x in this case exceed the given σ_y .

The problem is reduced to the sum of two solutions – the auxiliary problem and the main one. Both problems are reduced to infinite systems of linear algebraic equations, which allows the application of the reduction method to them. Basic solutions in different coordinate systems are combined using the analytical-numerical generalized Fourier method. This allowed obtaining a solution to the problem with a given accuracy.

The proposed solution method makes it possible to obtain the results of the stress-strain state for aircraft structures, building structures, technological machine-building lines, the loads on which are given by periodic functions.

In the future (in order to develop the specified research topic), it is necessary to consider models with cylindrical inclusions, as well as with several cylindrical cavities.

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Задача теорії пружності для шару з циліндричною порожниною за наявності періодичних навантажень

¹Т. М. Альошечкіна, ²Н. А. Українець, ²В. Ю. Мірошніков

¹Харківський національний університет міського господарства ім. О. М. Бекетова, 61002, Україна, м. Харків, вул. Черноглазівська, 17

²Національний аерокосмічний університет «Харківський авіаційний інститут», 61070, Україна, м. Харків, вул. Вадима Манька, 17

В аерокосмічній галузі й машинобудуванні використовуються елементи, навантаження на які описуються періодичними функціями. У задачах для шару з циліндричними неоднорідностями врахування таких навантажень є складним. З огляду на це існує необхідність створити методіку розрахунку напруженого стану для шару з циліндричною порожниною, за якою бралися до уваги й граничні умови у вигляді періодичної функції. У роботі запропоновано розв'язання задачі теорії пружності для шару з циліндричною порожниною у рамках узагальненого методу Фур'є. На верхній межі шару й на поверхні циліндричної порожнини задані напруження, а на нижній межі шару – переміщення. Шар і циліндрична порожнина розглядаються в різних системах координат (у декартовій та циліндричній). До рівнянь Ламе застосовуються функції перерозподілу узагальненого методу Фур'є. Задачу зведено до суми двох розв'язків – додаткової задачі й основної. Обидві задачі зведені до нескінченних систем лінійних алгебраїчних рівнянь, до яких допускається застосування методу редукції. Після знаходження невідомих у додатковій задачі обчислюється напруження в місці геометричного розташування порожнини. Основна задача розв'язується для шару з порожниною, на якій задані зі зворотнім знаком напруження, отримані з додаткової задачі. Повне рішення складається з додаткової та основної задач. Розрахувавши всі невідомі, можна отримати напружено-деформований стан у будь-якій точці тіла із заданою точністю. Чисельний аналіз напруженого стану показав високу точність виконання граничних умов і залежність від періодичного навантаження. Так, напруження σ_x та σ_z на верхній межі шару мають екстремуми в місцях максимальних значень σ_y і збільшуються їх від'ємні значення в місці розташування порожнини. Напруження σ_x при цьому перевищують задані σ_y .

Ключові слова: періодичне навантаження, шар з циліндричною порожниною, рівняння Ламе, узагальнений метод Фур'є.

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