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FIRST BASIC PROBLEM OF ELASTICITY THEORY FOR A LAYER WITH CYLINDRICAL CAVITIES SMOOTHLY CONTACTING TWO CYLINDRICAL BUSHINGS

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A spatial problem of elasticity is solved for a layer with n longitudinal cylindrical cavities, two of which contain thick-walled pipes in smooth contact with the layer. Stresses are given on the surfaces of the layer, the inner surfaces of the pipes, and the cavities. All canonical surfaces do not intersect each other. The material of the layer and cylindrical pipes is homogeneous and isotropic. An analytical and numerical calculation method, which assumes the fulfillment of statics conditions (for the first basic problem of elasticity theory) and is based on the Lamé equation, is proposed. The basic solutions of the Lamé equation are taken in a form that makes it possible to obtain an exact solution for a separate boundary surface in each separate coordinate system. The basic solutions in these coordinate systems (Cartesian for the layer and local cylindrical for the cylindrical inhomogeneities) are interconnected through the mathematical framework of the generalized Fourier method. The fulfillment of boundary conditions on the upper and lower surfaces of the layer, on the inner surfaces of pipes, on cylindrical cavities, as well as the consideration of interface conditions, create an infinite system of integro-algebraic equations, which is reduced to an infinite linear one. In the numerical study, the reduction method is applied to the resulting infinite linear algebraic system of equations. The solution of the system of equations gives the values of the unknown functions. Numerical calculations have shown the rapid convergence of approximate solutions to the exact one. The numerical analysis of the stressed state of the layer and thick-walled pipes showed that the use of polyamide bushings has almost no effect on the stress-strain state of the structure (compared to their absence), the use of steel bushings reduces the stress in the body of the layer in the areas of their location, redistributing the stress to the bushings themselves. The proposed solution method makes it possible to obtain the stress-strain state of structures containing cylindrical cavities and bushings, and the numerical analysis allows to assess the influence of the material on the values of stress distribution in the structures of machines and mechanisms at the design stage.

Keywords: *fiber composite, generalized Fourier method, Lamé equation, layer with cylindrical inclusions.*

Introduction

Bushings are important components in mechanical and aircraft engineering, they are used to reduce friction between moving parts, provide stability and alignment, help prevent wear and damage to critical structural components. Bushings are used in hinge joints of control surfaces to ensure smooth and controlled movement, to attach the engine to the airframe of an aircraft, reducing vibration and transferring loads. They are installed at the joints of rods and levers that transmit forces from the control units to the control surfaces.

They are also used to strengthen parts with holes. This is especially important in cases where the part is subjected to significant loads, and the hole weakens its structure. A bushing installed in a hole distributes the load over a larger area, reducing stress concentration around the hole. In some cases, bushings are installed to ensure a more accurate fit of connecting parts in holes. This is important in mechanisms where high positioning accuracy is required. Bushings are also used to repair damaged or worn holes.

In mechanical and aircraft engineering, high-strength and wear-resistant materials are used to manufacture bushings that can withstand extreme operating conditions, including high and low temperatures, significant loads and vibration, and are also able to reduce noise and increase comfort. The main materials include: steel, bronze, aluminum, Teflon, polyamide and other polymers.

Models of a number of structures can be represented in the form of a layer with cylindrical pipes and cavities. When designing such structures, it is necessary to have a distribution of stresses in their individual parts. However, computational models of such connections often turn out to be complex and cannot be calculated by classical analytical methods [1–3], which consider the problem either in a flat formulation or with a number of boundary conditions, not more than three.

Another approach that is often used to calculate these structures is numerical methods: the finite element method [4–6], the boundary element method [7], the finite difference method [8]. However, they are approximate and have a number of known drawbacks: the inability to take into account infinite elements, approximation errors, problems of convergence and stability of the calculation, significant computational resources, the inability to obtain an analytical expression that would describe the dependence of the solution on the parameters of the problem, sensitivity to the choice of initial conditions. It is also important to have a theoretical understanding of the problem or the results of field tests [9, 10] to interpret the numerical results. Analytical-numerical methods are hybrid approaches to solving problems and include both analytical (exact) and numerical (approximate) methods. These methods are often used to obtain more accurate results in complex problems where a purely analytical or purely numerical approach may be ineffective or too complex.

An example of a hybrid approach is the paper [11], where the displacements for the calculation of composite plates under impact loading are experimentally determined, after which the displacement vector parameters for each layer are expanded in a power series in the transverse coordinate. However, this approach cannot take into account longitudinal cylindrical inhomogeneities.

The solution of the problem of a layer with a perpendicularly located cylindrical cavity, presented in the papers [12, 13], is based on integral Laplace transforms and finite Fourier transforms. However, this method does not allow the analysis of systems with more than one cylindrical cavity.

In [14], metaheuristic algorithms and analytical methods were used to optimize thermal stresses in composite plates with non-circular holes, but this method cannot work with more than one cylindrical cavity.

The calculation of a layer with several cylindrical inhomogeneities located parallel to its boundaries is effectively carried out using the analytical-numerical generalized Fourier method [15], which is based on a combination of basic solutions of the Lamé equation in different coordinate systems. It has already been used to solve problems for a cylinder with cylindrical cavities [16, 17] and inclusions [18]. In [19], the justification of the formulas for the transition between the Cartesian and cylindrical coordinate systems is presented on the example of a half-space with a cylindrical cavity.

Problems for a layer with one cylindrical cavity using the generalized Fourier method in displacements are solved in [20], in stresses – in [21], or they're considered as mixed problems [22]. Problems for a layer with one pipe are solved in papers [23, 24]. However, the methods used in [20–24] do not allow solving the problem for several cylindrical inhomogeneities.

In paper [25], the method was developed and solved for a layer with two cylindrical cavities, and in [26] – with two cylindrical pipes under mixed boundary conditions. In paper [27], a solution of the first basic problem of the elasticity theory for a layer with two cylindrical pipes is proposed. However, the approach presented in [26, 27] does not allow taking into account additional cylindrical cavities.

Considering the abovementioned factors, the search for methods for solving problems for a layer with cylindrical bushings and cylindrical cavities is relevant.

The purpose of this paper is:

– creation of a method for solving the first fundamental problem of the elasticity theory (in stresses) for a layer smoothly contacting two longitudinal cylindrical tubes and weakened by n cylindrical cavities;

– determination of the stress state of a layer with one cylindrical cavity and two thick-walled pipes under balanced loading;

– determination of the influence of the presence of cylindrical bushings on the stressed-deformed state of the layer by conducting a comparative analysis with the option with cavities instead of thick-walled pipes [28].

Problem statement

The model under study consists of a layer in which two thick-walled cylindrical pipes and n cylindrical cavities are located parallel to its boundaries (Fig. 1). The pipes and cavities are considered in local cylindrical coordinates ρ_p, φ_p, z, where p=1 denotes the first pipe, p=2 – the second one, p=3…n+2 – cavity with number $p-2$. The layer is considered in the Cartesian coordinate system (x, y, z), which coincides with the coordinate system of the first pipe (p=1). The outer radii of the pipes and cavities are denoted by R_p. Internal pipe radii – r_p. Distance from the center of the Cartesian coordinate system to the layer boundaries is $y=h$ and $y=-\tilde{h}$.

The stresses are given: at the upper and lower boundaries of the layer

$$F\vec{U}(x,z)\big|_{y=h} = \vec{F}_h^0(x,z); \quad F\vec{U}(x,z)\big|_{y=-\tilde{h}} = \vec{F}_{\tilde{h}}^0(x,z),$$

where
$$\vec{F}_h^0(x,z)=\tau_{yx}^{(h)}\cdot\vec{e}_x+\sigma_y^{(h)}\cdot\vec{e}_y+\tau_{yz}^{(h)}\cdot\vec{e}_z\,;\ \vec{F}_{\tilde{h}}^0(x,z)=\tau_{yx}^{(\tilde{h})}\cdot\vec{e}_x+\sigma_y^{(\tilde{h})}\cdot\vec{e}_y+\tau_{yz}^{(\tilde{h})}\cdot\vec{e}_z\,; \tag{1}$$
on the inner surfaces of the pipes the following stresses are given
$$F\vec{U}_0^{(p)}(\varphi_p,z)\big|_{\rho_3=R}=\sigma_p^{(p)}\vec{e}_p+\tau_{\rho\varphi}^{(p)}\vec{e}_\varphi+\tau_{\rho z}^{(p)}\vec{e}_z\,;\ p=1,\,2; \tag{2}$$
on the surface of cylindrical cavities
$$F\vec{U}(\varphi_p,z)\big|_{\rho p=R_p}=\vec{F}_p^0(\varphi_p,z)=\sigma_\rho^{(p)}\vec{e}_\rho+\tau_{\rho\varphi}^{(p)}\vec{e}_\varphi+\tau_{\rho z}^{(p)}\vec{e}_z\,;\ p=3,\,\ldots,\,n+2, \tag{3}$$
where n is the number of cavities; $\vec{U}$ is the displacement in a layer;
$F\vec{U}=2G\left[\dfrac{\sigma}{1-2\sigma}\vec{n}\cdot\overrightarrow{divU}+\dfrac{\partial}{\partial n}\vec{U}+\dfrac{1}{2}\left(\vec{n}\times\overrightarrow{rotU}\right)\right]$ is the stress operator.

The layer is connected to each pipe via interface conditions
$$\left.\begin{aligned}
&U_{0,\rho}(\varphi_p,z)\big|_{\rho_p=R_p}=U_{p,\rho}(\varphi_p,z)\big|_{\rho_p=R_p}\\
&\sigma_{0,\rho}(\varphi_p,z)\big|_{\rho_p=R_p}=\sigma_{p,\rho}(\varphi_p,z)\big|_{\rho_p=R_p}\\
&\tau_{0,\rho\varphi}(\varphi_p,z)\big|_{\rho_p=R_p}=0\\
&\tau_{p,\rho\varphi}(\varphi_p,z)\big|_{\rho_p=R_p}=0\\
&\tau_{0,\rho z}(\varphi_p,z)\big|_{\rho_p=R_p}=0\\
&\tau_{p,\rho z}(\varphi_p,z)\big|_{\rho_p=R_p}=0
\end{aligned}\right\}, \tag{4}$$

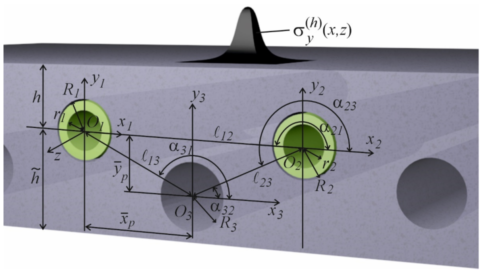

Fig. 1. Layer with two cylindrical thick-walled pipes and cylindrical cavities

where $U_{0,\rho}(\varphi_p,z)$ are normal displacements for the layer; τ_0 are – tangential stresses for the layer; $U_{p,\rho}(\varphi_p,z)$ are normal displacements for pipes; τ_p are tangential stresses for pipes.

In this case, based on the static conditions, the equilibrium equations must be satisfied
$$\iint_{(\sigma)}\vec{F}(M)d\sigma=0\,;\quad \iint_{(\sigma)}\vec{r}\times\vec{F}(M)d\sigma=0\,,$$
where $\sigma=\{\sigma_1\cup\sigma_2\cup\sigma_3\}$; σ_1 is the plane at $y=h$; σ_2 is the plane at $y=-\tilde{h}$; σ_3 is the cylinder surface $\rho=R_p$,
$\vec{F}M=\begin{cases}\vec{F}_h^0(x,z)\text{ at }\sigma_1\\ \vec{F}_{\tilde{h}}^0(x,z)\text{ at }\sigma_2\\ \vec{F}_R^0(\varphi,z)\text{ at }\sigma_3\end{cases}$; $\vec{r}$ is the radius vector of p. M.

All the given functions are assumed to decrease at infinity.

Solution methodology

We seek the solution of the Lamé equations in the form
$$\vec{U}_0=\sum_{p=1}^{n+2}\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}B_{k,m}^{(p)}(\lambda)\cdot\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)d\lambda+$$
$$+\sum_{k=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(H_k(\lambda,\mu)\cdot\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)+\tilde{H}_k(\lambda,\mu)\cdot\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)\right)d\mu\,d\lambda\,; \tag{5}$$
$$\vec{U}_i=\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}A_{k,m}^{(i)}(\lambda)\cdot\vec{R}_{k,m}(\rho_1,\varphi_1,z;\lambda)+\tilde{A}_{k,m}^{(i)}(\lambda)\cdot\vec{S}_{k,m}(\rho_1,\varphi_1,z;\lambda)\,d\lambda\,, \tag{6}$$

where $\vec{U}_i$ is the displacement in the i-th pipe; $H_k(\lambda,\mu)$, $\widetilde{H}_k(\lambda,\mu)$, $B_{k,m}^{(p)}(\lambda)$, $A_{k,m}^{(i)}(\lambda)$, $\widetilde{A}_{k,m}^{(i)}(\lambda)$ are unknown functions that need to be found from the boundary conditions (1) – (3) and the interface conditions (4); $\vec{S}_{k,m}(\rho_1,\varphi_1,z;\lambda)$, $\widetilde{R}_{k,m}(\rho_1,\varphi_1,z;\lambda)$, $\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)$, $\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)$ are basic solutions of the Lamé equation, which are given in the form [15]:

$$\vec{u}_k^{\pm}(x,y,z;\lambda,\mu)=N_k^{(d)}e^{i(\lambda z+\mu x)\pm\gamma y};$$

$$\vec{R}_{k,m}(\rho,\varphi,z;\lambda)=N_k^{(p)}I_m(\lambda\rho)e^{i(\lambda z+m\varphi)};\ \vec{S}_{k,m}(\rho,\varphi,z;\lambda)=N_k^{(p)}\left[(sign\lambda)^m K_m(|\lambda|\rho)\cdot e^{i(\lambda z+m\varphi)}\right];\ k=1,2,3;$$

$$N_1^{(d)}=\frac{1}{\lambda}\nabla;\ N_2^{(d)}=\frac{4}{\lambda}(\nu-1)\vec{e}_2^{(1)}+\frac{1}{\lambda}\nabla(y\cdot);\ N_3^{(d)}=\frac{i}{\lambda}\operatorname{rot}\left(\vec{e}_3^{(1)}\cdot\right);$$

$$N_1^{(p)}=\frac{1}{\lambda}\nabla;\ N_2^{(p)}=\frac{1}{\lambda}\left[\nabla\left(\rho\frac{\partial}{\partial\rho}\right)+4(\nu-1)\left(\nabla-\vec{e}_3^{(2)}\right)\frac{\partial}{\partial z}\right];\ N_3^{(p)}=\frac{i}{\lambda}\operatorname{rot}\left(\vec{e}_3^{(2)}\cdot\right);\ \gamma=\sqrt{\lambda^2+\mu^2};\ -\infty<\lambda,\mu<\infty,$$

where ν is the Poisson's; $I_m(x)$, $K_m(x)$ are modified Bessel functions.

To write equations (5) and (6) in the same coordinate system, the transition formulas between the basic solutions of the Lamé equation [11] were applied:

– from the external solutions for the cylinder $\vec{S}_{k,m}$ to solutions for the layer $\vec{u}_k^{(-)}$ (at $y>0$) and $\vec{u}_k^{(+)}$ (at $y<0$)

$$\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)=\frac{(-i)^m}{2}\int_{-\infty}^{\infty}\omega_{\mp}^m\cdot e^{-i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\vec{u}_k^{(\mp)}\cdot\frac{d\mu}{\gamma},\ k=1,3;$$

$$\vec{S}_{2,m}(\rho_p,\varphi_p,z;\lambda)=\frac{(-i)^m}{2}\int_{-\infty}^{\infty}\omega_{\mp}^m\cdot\left(\left(\pm m\cdot\mu-\frac{\lambda^2}{\gamma}\pm\lambda^2\bar{y}_p\right)\vec{u}_1^{(\mp)}\mp\lambda^2\vec{u}_2^{(\mp)}\pm4\mu(1-\sigma)\vec{u}_3^{(\mp)}\right)\cdot\frac{e^{-i\mu\bar{x}_p\pm\gamma\bar{y}_p}d\mu}{\gamma^2}\quad(7)$$

where $\gamma=\sqrt{\lambda^2+\mu^2}$, $\omega_{\mp}(\lambda,\mu)=\frac{\mu\mp\gamma}{\lambda}$, $m=0,\pm1,\pm2,\ldots$;

– from solutions for the layer $\vec{u}_k^{(+)}$ and $\vec{u}_k^{(-)}$ to internal solutions for the cylinder $\vec{R}_{k,m}$

$$\vec{u}_k^{(\pm)}(x,y,z)=e^{i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\sum_{m=-\infty}^{\infty}(i\cdot\omega_{\mp})^m\vec{R}_{k,m},\ k=1,3;$$

$$\vec{u}_2^{(\pm)}(x,y,z)=e^{i\mu\bar{x}_p\pm\gamma\bar{y}_p}\cdot\sum_{m=-\infty}^{\infty}\left[(i\cdot\omega_{\mp})^m\cdot\lambda^{-2}\left(\left(m\cdot\mu+\bar{y}_p\cdot\lambda^2\right)\cdot\vec{R}_{1,m}\pm\gamma\cdot\vec{R}_{2,m}+4\mu(1-\sigma)\vec{R}_{3,m}\right)\right],\quad(8)$$

where $\vec{R}_{k,m}=\vec{\tilde{b}}_{k,m}(\rho_p,\lambda)\cdot e^{i(m\varphi_p+\lambda z)}$; $\vec{\tilde{b}}_{1,n}(\rho,\lambda)=\vec{e}_\rho\cdot I_n'(\lambda\rho)+i\cdot I_n(\lambda\rho)\cdot\left(\vec{e}_\varphi\frac{n}{\lambda\rho}+\vec{e}_z\right)$;

$$\vec{\tilde{b}}_{2,n}(\rho,\lambda)=\vec{e}_\rho\cdot\left[(4\sigma-3)\cdot I_n'(\lambda\rho)+\lambda\rho I_n''(\lambda\rho)\right]+\vec{e}_\varphi i\cdot m\left(I_n'(\lambda\rho)+\frac{4(\sigma-1)}{\lambda\rho}I_n(\lambda\rho)\right)+\vec{e}_z i\lambda\rho I_n'(\lambda\rho);$$

$\vec{\tilde{b}}_{3,n}(\rho,\lambda)=-\left[\vec{e}_\rho\cdot I_n(\lambda\rho)\frac{n}{\lambda\rho}+\vec{e}_\varphi\cdot i\cdot I_n'(\lambda\rho)\right]$; $\vec{e}_\rho$, $\vec{e}_\varphi$, $\vec{e}_z$ are orts in a cylindrical coordinate system;

– from solutions for the cylinder with number p to solutions for the cylinder with number q

$$\vec{S}_{k,m}(\rho_p,\varphi_p,z;\lambda)=\sum_{n=-\infty}^{\infty}\vec{b}_{k,pq}^{mn}(\rho_q)\cdot e^{i(n\varphi_q+\lambda z)},\ k=1,2,3;$$

$$\vec{b}_{1,pq}^{mn}(\rho_q)=(-1)^n\widetilde{K}_{m-n}(\lambda\ell_{pq})\cdot e^{i(m-n)\alpha_{pq}}\cdot\vec{\tilde{b}}_{1,n}(\rho_q,\lambda);\ \vec{b}_{3,pq}^{mn}(\rho_q)=(-1)^n\widetilde{K}_{m-n}(\lambda\ell_{pq})\cdot e^{i(m-n)\alpha_{pq}}\cdot\vec{\tilde{b}}_{3,n}(\rho_q,\lambda);$$

$$\vec{b}_{2,pq}^{mn}(\rho_q)=(-1)^n\left\{\widetilde{K}_{m-n}(\lambda\ell_{pq})\cdot\vec{\tilde{b}}_{2,n}(\rho_q,\lambda)-\frac{\lambda}{2}\ell_{pq}\cdot\left[\widetilde{K}_{m-n+1}(\lambda\ell_{pq})+\widetilde{K}_{m-n-1}(\lambda\ell_{pq})\right]\cdot\vec{\tilde{b}}_{1,n}(\rho_q,\lambda)\right\}\cdot e^{i(m-n)\alpha_{pq}},\quad(9)$$

where α_{pq} is the angle between the x_p axis and the segment ℓ_{qp}; $\widetilde{K}_m(x)=(sign(x))^m\cdot K_m(|x|)$.

Satisfying the boundary conditions, a system of integro-algebraic equations was formed to find the unknown functions (5) and (6). Given the cumbersomeness of the specified system [29], it is omitted in this paper.

The first six equations were derived by applying the boundary conditions to the flat surfaces of the layer (1). For this, the stress operator was applied to the functions (5), and the double Fourier integral was applied to equation (1), after which the obtained expressions were equated. Basic solutions $\vec{S}_{k,m}$, given in a cylindrical coordinate system, were transformed into a Cartesian system $\vec{u}_k^{\pm}$ using the transition formulas (7).

Six additional equations arise when applying boundary conditions to the inner surfaces of the pipes (2). For this, the stress operator is applied to the functions (6), and the resulting expressions are equated to the functions (2), to which the Fourier integral along the z axis and the Fourier series along the angle φ are previously applied.

Three boundary condition equations are written for each cavity surface: the stress operator is applied to the functions (5), and the integral and Fourier series are applied to (3), after which they are equated to each other. Basic solutions $\vec{u}_k^{\pm}$ from the Cartesian coordinate system, using the transition formulas (8), rewritten as $\vec{R}_{k,m}$ into the cylindrical one.

An additional 12 equations arise from the interface conditions between the layer and each of the two pipes. To apply these conditions, the basic solutions $\vec{u}_k^{\pm}$, given in a Cartesian coordinate system, are transformed into solutions $\vec{R}_{k,m}$ of the local cylindrical systems using the transition functions (8). The transition formulas (9) are also used for basic solutions between different local cylindrical coordinate systems.

Using the first six equations, we have expressed $H_k(\lambda,\mu)$ and $\widetilde{H}_k(\lambda,\mu)$ through $B_{k,m}^{(p)}(\lambda)$ and substituted them into other equations. After simplifying the expressions, getting rid of the series and integrals (which are now the same on the right and left sides of each equation), we obtained an infinite system of $(12+n\cdot 3)$ linear algebraic equations of the second kind, which was solved by the reduction method. As a result, the unknowns $B_{k,m}^{(p)}(\lambda)$, $A_{k,m}^{(1)}(\lambda)$, $\widetilde{A}_{k,m}^{(1)}(\lambda)$, $A_{k,m}^{(2)}(\lambda)$, $\widetilde{A}_{k,m}^{(2)}(\lambda)$ were found. Next, substituting the obtained values $B_{k,m}^{(p)}(\lambda)$ in expressions for $H_k(\lambda,\mu)$ and $\widetilde{H}_k(\lambda,\mu)$, we have identified all the unknown problems.

Numerical studies of the stressed state

The problem is numerically solved for a layer with two cylindrical pipes and one cylindrical cavity at a given balanced load (Fig. 2).

Geometric parameters: pipes and cylindrical cavity are located on the same horizontal axis ($\alpha_{12}=0$, $\alpha_{13}=0$), distance between pipes is $\ell_{12}=100$ mm, distance to cylindrical cavity is $L_{13}=50$ mm, outer radius of pipes and cylindrical cavity is $R_1=R_2=R_3=15$ mm, internal radius is $r_1=r_2=10$ mm, distances to the upper and lower boundaries of the layer are $h=\widetilde{h}=25$ mm.

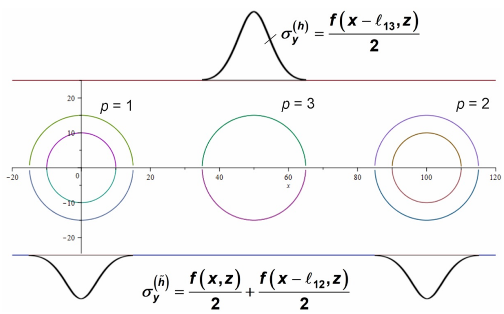

Fig. 2. Calculation model of the structure with bushings

Physical characteristics of the layer: aluminum plate D16T, Poisson's ratio $\nu_0=0.3$, elasticity modulus $E_0=7.1\times10^4$ MPa. The physical characteristics of the pipes were calculated in two options: steel ShKh15 (Poisson's ratio $\nu_1=\nu_2=0.28$, elasticity modulus $E_1=E_2=2.16\times10^5$ MPa) and polyamide ($\nu_1=\nu_2=0.4$, elasticity modulus $E_1=E_2=1.65\times10^3$ MPa). The stresses, for comparison, were also obtained for a layer with three cylindrical cavities (without bushings) [28].

The stresses are given:

– at the upper boundary of the layer

$$\sigma_y^{(h)}(x,z) = \frac{F\cdot 10^8}{(z^2+10^2)^2\cdot\left((x-\ell_{13})^2+10^2\right)^2};\ \tau_{yx}^{(h)}=\tau_{yz}^{(h)}=0;\ F=1;$$

– at the bottom edge of the layer

$$\sigma_y^{(\tilde{h})}(x,z)=\frac{F-10^8}{2(z^2+10^2)^2\cdot\left(x^2+10^2\right)^2}+\frac{F-10^8}{2(z^2+10^2)^2\cdot\left((x-\ell_{12})^2+10^2\right)^2};\ \tau_{yx}^{(\tilde{h})}=\tau_{yz}^{(\tilde{h})}=0\,.$$

Zero stresses are given:

– on the inner surfaces of the pipes $\sigma_\rho^{(p)}(\varphi_p,z)=\tau_{\rho\varphi}^{(p)}\cdot\vec{e}_\varphi=\tau_{\rho z}^{(p)}=0$, p=1, 2;

– on the surface of the cavity $\sigma_\rho^{(3)}=\tau_{\rho\varphi}^{(3)}=\tau_{\rho z}^{(3)}=0$.

The infinite system was truncated by the parameter m=6 (the number of terms in the Fourier series and the order of the system of equations).

The accuracy of fulfilling the boundary conditions for the specified m and the specified geometric parameters is not less than 10^{-4} for values from 0 to 1.

The stresses σ_φ on the inner and outer surfaces of the pipe p=1 at z=0 are shown in Fig. 3. The stresses σ_φ on the pipe surface p=2 are mirrored.

The stresses σ_φ in the middle of the bushings depend significantly on the material of these bushings (Fig. 3). If the bushings are made of polyamide, the stresses σ_φ in them are almost zero.

When making bushings from steel, the stresses σ_φ are not equal to zero, and the maximum values occur on the inner surface of this bushing (Fig. 3, line 1).

The inner and outer surfaces of the bushings have different sign values of stresses σ_φ (Fig. 3, lines 1, 2). This means that the bushing (along the thickness of the flange) works in bending.

The stresses σ_φ in the middle of the layer on the surfaces of cylindrical holes at z=0 are shown in Fig. 4.

The stresses σ_φ in the middle of the layer on the surface of the cylindrical cavity p=3 (Fig. 4, line1) do not depend on the material of the bushings: $\sigma_\varphi^{(\text{steel})}=\sigma_\varphi^{(\text{polyamide})}=\sigma_\varphi^{(\text{cavity})}$.

The stresses σ_φ on the surface of the hole p=1 depend on the material of the bushing. If the bushing is made of polyamide, then the stresses σ_φ on the surface of the hole p=1 are equal to the stresses σ_φ without the bushings (Fig. 4, line 3). If the bushing is made of steel, then the stresses σ_φ on the surface of the hole p=1 (Fig. 4, line 2) are almost two times less than the stresses σ_φ with the polyamide bushing.

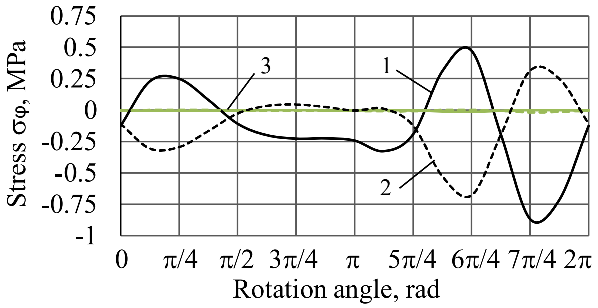

Fig. 3. Stresses σ_φ on the surfaces of pipes:
1 – pipe (steel), ρ=r_1 ; 2 – pipe (steel), ρ=R_1;
3 – pipe (polyamide), ρ=R_1

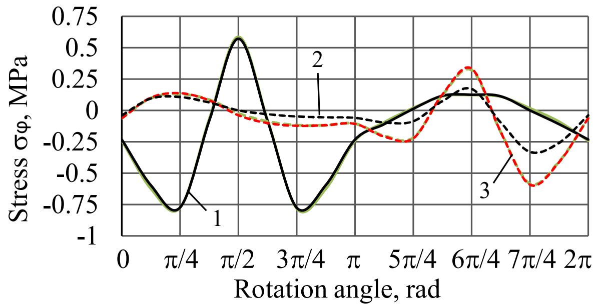

Fig. 4. Stresses σ_φ in the middle of the layer on the surfaces of the holes:
1 – cavity p=3; 2 – hole p=1, steel;
3 – hole p=1, polyamide

The stresses σ_z in the middle of the layer on the surfaces of cylindrical holes at z=0 are shown in Fig. 5.

The stresses σ_z in the middle of the layer on the surface of the cylindrical cavity p=3 (Fig. 5, line 1) do not depend on the material of the bushings: $\sigma_z^{(\text{steel})}=\sigma_z^{(\text{polyamide})}=\sigma_z^{(\text{cavity})}$.

If the bushing is made of polyamide, then the stresses σ_z on the surface of the hole p=1 (Fig. 5, line 3) differ little from the stresses σ_z without the bushing (Fig. 5, line 4) and more than double the stresses σ_z if the bushing is made of steel (Fig. 5, line 2).

The stresses σ_x on the upper and lower surfaces of the layer along the x axis at z=0 are shown in Fig. 6.

The stresses σ_x along the x axis at the upper boundary of the layer are almost independent of the material of the bushing or its absence (Fig. 6, line 1). The maximum values on this surface occur above the cylindrical cavity (p=3) and exceed the specified normal stresses σ_y.

At the lower boundary of the layer, the stresses σ_x along the x axis, depending on the material of the bushings, change in the region of these inhomogeneities (Fig. 6, lines 2, 3).

The stresses σ_x along the x axis at the lower boundary of the layer, in the absence of bushings, are equal to the stresses σ_x when the material of the bushings is polyamide (Fig. 6, line 3).

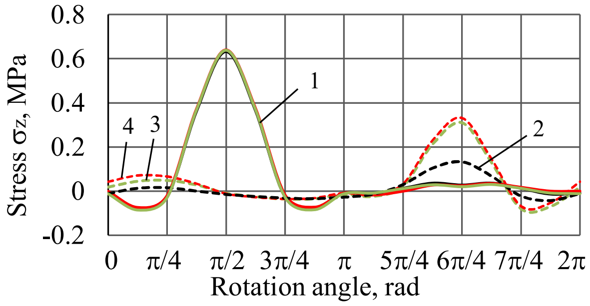

Fig. 5. Stresses σ_z in the middle of the layer on the surfaces of the holes:
1 – cavity p=3; 2 – hole p=1, steel;
3 – hole p=1, polyamide; 4 – hole p=1, cavity

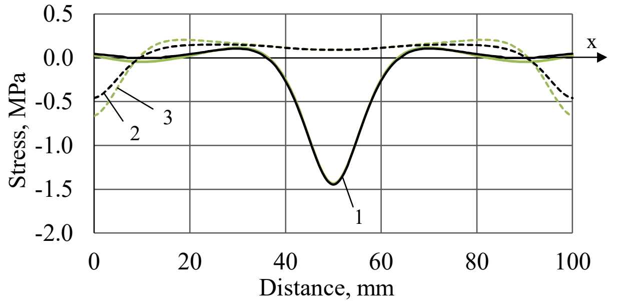

Fig. 6. Stresses σ_x on flat surfaces of the layer:
1 – on the upper boundary ($y=h$);
2 – on the lower boundary ($y=-\tilde{h}$), steel bushing;
3 – on the lower boundary ($y=-\tilde{h}$), polyamide bushing

Conclusions

1. An analytical-numerical approach to solving the first basic spatial problem of the theory of elasticity for a layer smoothly contacting two longitudinal cylindrical tubes and weakened by n cylindrical cavities is proposed.

2. The problem is reduced to an infinite system of linear algebraic equations of the second kind, which made it possible to apply the reduction method. The use of the analytical-numerical generalized Fourier method ensured obtaining a solution with the required accuracy.

3. The stress state of a layer with one cylindrical cavity and two thick-walled tubes under balanced loading is numerically determined.

4. A comparative analysis of the stress state of the layer for an option with cavities instead of thick-walled tubes [28] is carried out.

5. The numerical analysis of the stress state of the layer and thick-walled pipes under balanced loading showed that the use of:

– polyamide bushings, compared to their absence, has almost no effect on the stress-strain state of the structure;

– steel bushings reduces the stress in the middle of the layer in the areas of their location, redistributing the stress to the bushings themselves.

The proposed solution method makes it possible to obtain the stress-strain state of structures containing cylindrical cavities and bushings.

The numerical analysis allows to assess the influence of the material on the magnitude of the stress distribution in the structures of machines and mechanisms at the design stage.

In the further development of the specified research topic, it is necessary to consider models with other boundary conditions. One of such options is to take into account the rigid connection between the layer and pipes.

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Перша основна задача теорії пружності для шару з циліндричними порожнинами й гладко контактуючого з двома циліндричними втулками

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Розв'язана просторова задача теорії пружності для шару з n поздовжніми циліндричними порожнинами, дві з яких містять товстостінні труби, гладко контактуючі з шаром. На поверхнях шару, внутрішніх поверхнях труб і порожнинах задані напруження. Усі канонічні поверхні між собою не перетинаються. Матеріал шару й циліндричних труб однорідний та ізотропний. Запропоновано аналітико-числову методику розрахунку, яка передбачає виконання умов статики (для першої основної задачі теорії пружності) та базується на рівнянні Ламе. Базисні розв'язки рівняння Ламе беруться у формі, що дає можливість у кожній окремій системі координат отримати точний розв'язок для певної граничної поверхні. Базисні розв'язки в цих системах координат (для шару – декартова, для циліндричних неоднорідностей – локальні циліндричні) пов'язані між собою математичним апаратом узагальненого методу Фур'є. Виконання граничних умов на верхній та нижній поверхнях шару, на внутрішніх поверхнях труб, на циліндричних порожнинах, а також врахування умов спряження створюють нескінчену систему інтегро-алгебраїчних рівнянь, яка зведена до нескінченої лінійної. У чисельному дослідженні до отриманої нескінченої лінійної алгебраїчної системи рівнянь застосовується метод редукції. Розв'язання системи рівнянь дає значення невідомих функцій. Чисельні розрахунки показали швидку збіжність наближених розв'язків до точного. Проведений чисельний аналіз напруженого стану шару і товстостінних труб показав, що застосування поліамідних втулок, в порівнянні з їх відсутністю, майже не впливає на напружено-деформований стан конструкції, застосування стальних втулок зменшує напруження в середині шару в областях їх розташування, перерозподіляючи напруження на самі втулки. Запропонований метод розв'язання дає можливість отримувати напружено-деформований стан конструкцій, що містять циліндричні порожнини і втулки, а проведений чисельний аналіз дозволяє оцінити вплив матеріалу на величини розподілення напружень у конструкціях машин і механізмів на етапі проєктування.

Ключові слова: *волокнистий композит, узагальнений метод Фур'є, рівняння Ламе, шар з циліндричними включеннями.*

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