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ANALYSIS OF DAMPING OF FLUID OSCILLATIONS IN SPHERICAL TANKS USING THE BOUNDARY ELEMENT METHOD

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The aim of this study is to develop numerical methods for analyzing the stability of fluid motion in spherical tanks with horizontal baffles. Partially filled spherical tanks are important components of modern engineering systems. They are widely used as storage vessels for drinking water and hazardous liquids, as well as structural elements of launch vehicle fuel tanks. Experimental testing of such tanks for strength and dynamic stability is generally expensive and not always safe. This necessitates the development of virtual testing methods based on efficient computational algorithms. In this context, the development of new numerical methods for analyzing fluid oscillations and motion stability in tanks, where the radius of the free surface depends on the filling level, is a relevant problem. The study employs methods of potential theory, the boundary element method, the method of prescribed normal modes, and numerical techniques for solving systems of differential equations. Spectral boundary value problems are solved to determine the natural frequencies and mode shapes of fluid oscillations in spherical tanks without baffles and in tanks with horizontal baffles containing openings of various diameters. These problems are reduced to systems of one-dimensional singular integral equations. The obtained natural modes are used as basis functions for solving the problem of forced fluid oscillations in spherical tanks subjected to simultaneous vertical and horizontal excitations. Expressions for the velocity potential and the free surface elevation function are derived in the form of infinite series, and the convergence of these series is analyzed. The problem of determining the dynamic characteristics of the fluid is reduced to solving a system of ordinary differential equations of the Mathieu type, which makes it possible to study the stability of fluid motion in a spherical tank under combined horizontal and vertical loading. An efficient numerical approach for studying fluid oscillations and motion stability in partially filled spherical tanks has been developed and implemented. The proposed approach can be used for virtual testing of spherical tanks and for analyzing fluid behavior in the design and operation of tanks and fuel systems in aerospace engineering.

Keywords: *fluid oscillations, damping, spherical tanks, horizontal baffles, singular integral equations, boundary element method, stability of fluid motion.*

Introduction

Shell systems and shell-type tanks with compartments filled with fluid are widely used in various branches of modern industry, in particular, in aerospace, energy, oil and gas, and transport. Such objects include, for example, fuel tanks of launch vehicles, aircrafts and spacecrafts, sea vessels, as well as ground-based tanks for water storing, petroleum products, liquefied gases, and other fluids. During operation, such tanks, which may contain hazardous or aggressive media, are often subjected to serious loads, including seismic loads. If the tank is only partially filled, under the influence of intense disturbances, the phenomenon of fluid splashing occurs - low-frequency oscillations of its free surface. As a result, significant local hydrodynamic pressures can act on the tank walls, which can lead to loss of structural stability, deviation of the trajectory of spacecraft, or even large-scale environmental consequences in the event of tank destruction during earthquakes or terrorist acts. Such oscillations become especially dangerous at the final stages of operation, when the fluid level decreases, and a significant free volume remains in the tank, which contributes to the development of intense splashing. To reduce the amplitude of these oscillations, various constructive means are used, in particular, internal baffles and special coatings of the fluid free surface. However, experimental studies of such damping devices are quite expensive and may be accompanied by damage to expensive test specimens or undesirable environmental consequences. In this regard, computer modeling of fluid dynamics in tanks in motion becomes important. Such computer studies make it possible to perform virtual tests of structural elements, find destructive loads, optimize the design parameters of tanks with various types of dampers, and increase their reliability in real operating conditions.

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Overview of the current state of the issue

At the initial stages of studying the stability of partially filled tanks, relatively simple mathematical models and analytical approaches were used [1–2]. Despite their simplified nature, these methods made it possible to clarify the peculiarities of the nature of the fluid motion in tanks and cisterns, as well as to outline the possibilities of optimizing their design parameters. Further development of this direction led to the use of spectral analysis aimed at determining the lowest (fundamental) natural frequencies of fluid oscillations in rigid tanks within the linear formulation of the problem [3]. Subsequently, these models were improved by taking into account the elastic properties of the tank walls [4–5].

Recently, modern numerical approaches have been actively used to study the strength and oscillation processes in structures interacting with a fluid. These include the finite element method (FEM) [6], the boundary element method (BEM) [7], meshless methods [8], and series expansion methods [9]. In addition, various effective means of reducing the amplitude of fluid splashes have been proposed, in particular, the use of horizontal and vertical baffles [10–11] and floating covers [12]. The effectiveness of such technical solutions has been confirmed by both numerical and experimental studies. Fluid splashes in prismatic tanks were studied in [13], while elliptical tanks were considered in [14]. A quantitative assessment of the effect of baffles on reducing the instability caused by fluid splashes is given in [15]. Experimental studies [16] performed on tank models with one degree of freedom revealed a significant effect of hydrodynamic damping under conditions of vertical excitation. Thanks to comprehensive studies combining theoretical and experimental methods, significant progress in understanding the mechanisms of dynamic stability of shell structures under dynamic loads has been achieved [17]. In papers [18–19], a dynamic analysis of elastic tanks filled with compressible (acoustic) fluids was carried out in a linear formulation taking into account splashing and capillary effects on the free surface.

In this paper, the authors present improved numerical methods for solving spectral boundary value problems to determine the fundamental frequencies and forms of fluid oscillations in a rigid shell of rotation. The obtained forms are used as basis functions in the study of forced fluid oscillations in a tank under the action of combined horizontal and vertical loads.

Research objective and problem statement

The aim of the study is to develop a numerical method for studying the stability of fluid motion in tanks taking into account the presence of a horizontal baffle.

In [10, 20] it is shown that the lowest natural frequencies of the shell-fluid system correspond to the modes of oscillations of the fluid free surface in a rigid shell. In [20] conditions under which the interaction between the oscillations of elastic walls and the splashing of the free surface cannot be considered insignificant are also determined; however, such conditions

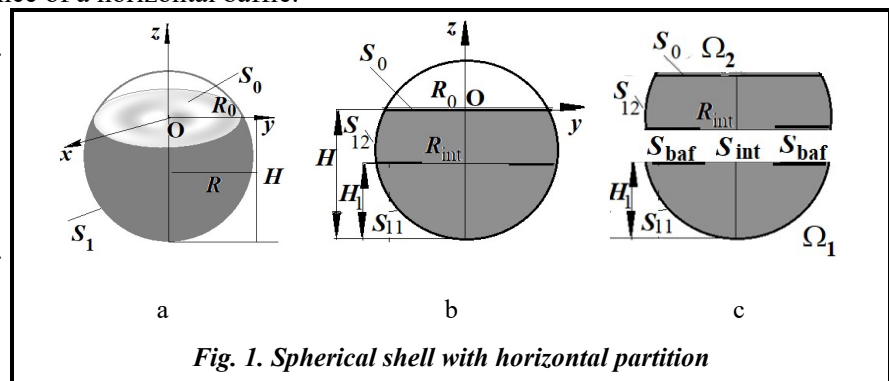


Fig. 1. Spherical shell with horizontal partition

exist only for shells of sufficiently small thickness. In view of this, rigid shells partially filled with fluid are considered further (Fig. 1, a).

A spherical shell with a horizontal baffle is studied (Fig. 1), its influence on the spectrum of natural frequencies and the shape of oscillations of the coupled shell-fluid system, as well as on the magnitude of the free surface elevation under the action of combined harmonic loads in the horizontal and vertical directions, is analyzed.

S_1 denotes the wetted surface of the shell, and S_0 denotes the free fluid surface. It should be noted that the surface S_1 includes the walls S_{11} and S_{12} , as well as the baffle surface S_{baf} , installed in the shell at a height H_1 . The fluid is assumed to be inviscid, incompressible, and its motion is vortex-free; the influence of capillary effects is neglected. Since the flow is vortex-free, there exists a scalar velocity potential Φ such that $V = \nabla\Phi$, where $V=(V_1, V_2, V_3)$ – velocity vector. In the domain $\Omega=\{0 \leq r \leq R, -H \leq z \leq 0\}$, which is occupied by the fluid, this potential satisfies the Laplace equation $\nabla^2\Phi(x, t) = 0$. For this equation, we set the following boundary conditions:

$$\frac{\partial \Phi}{\partial n} \Big|_{S_1} = 0; \quad \frac{\partial \Phi}{\partial n} \Big|_{S_0} = \frac{\partial \zeta}{\partial t}; \quad \frac{p - p_0}{\rho_l} = -\frac{\partial \Phi}{\partial t} - (g + a_v(t))\zeta + a_h(t)x \Big|_{S_0} = 0,$$

where n is the unit external normal to the surface; p_0 is the atmospheric pressure; p is the fluid pressure; $a_v(t), a_h(t)$ is the acceleration due to the applied force in the vertical and horizontal directions, respectively; $\zeta = \zeta(x, y, t)$ is an unknown function that describes the change in the level of the fluid free surface with time.

Thus, the study of fluid splashes in a rigid shell is reduced to determining two unknown functions, Φ and ζ , by solving such a boundary value problem

$$\nabla^2 \Phi = 0; \quad \frac{\partial \Phi}{\partial n} \Big|_{S_1} = 0; \quad \frac{\partial \Phi}{\partial n} \Big|_{S_0} = \frac{\partial \zeta}{\partial t}; \quad p - p_0 \Big|_{S_0} = 0.$$

The mode superposition method is widely used to solve coupled boundary value problems of dynamics [20], in particular, problems of interaction of structures with fluid [10]. Its efficiency is due to the transition to generalized coordinates, which allows to significantly reduce computational costs. Within the normal mode approach, unknown functions Φ and ζ for shells of rotation are given as expansions in their own forms

$$\Phi(x, y, z, t) = \sum_{k=1}^n \dot{d}_k(t) \varphi_k(x, y, z); \quad \zeta(x, y, t) = \sum_{k=1}^n d_k(t) \zeta_k(x, y),$$

where $\varphi_k(x, y, z)$ and $\zeta_k(x, y)$ are basis functions; $d_k(t)$ and $\dot{d}_k(t)$ are generalized coordinates and their time derivatives. Functions φ_k and ζ_k satisfy the following spectral boundary value problems [10]:

$$\nabla^2 \varphi_k = 0; \quad \frac{\partial \varphi_k}{\partial n} \Big|_{S_1} = 0; \quad \frac{\partial \varphi_k}{\partial n} = \frac{\chi_k^2}{g} \varphi_k \Big|_{S_1}, \tag{1}$$

where χ_k are natural frequencies corresponding to natural forms φ_k . It is assumed that the position of the free surface at the initial time corresponds to $z=0$. Then, according to (1), the free surface elevation $\zeta(x, y, t)$ can be written as

$$\zeta(x, y, t) = \frac{1}{g} \sum_{k=1}^n \chi_k^2 d_k(t) \varphi_k(x, y, 0).$$

Thus, the basis functions for calculating both Φ and ζ can be defined using functions $\varphi_k(x, y, z)$. It should be noted that the boundary value problem (1) is a spectral boundary value problem [20]. Its solution gives a system of basis functions, which is further used to model the fluid motion in tanks in both linear and nonlinear formulations, under the action of external forces that perturb.

Integral equation method

A set of basis functions $\{\varphi_k(x, y, z)\}$, which are used to approximate the forms of fluid splashing in shells of rotation is determined by solving a system of singular integral equations. The specified system is formulated within the conceptual approach proposed in [21–22]. To simplify the notation, we further omit the index k ; then the main integral relation that defines the function $\varphi_k(x, y, z)$ can be submitted in the following form

$$2\pi\varphi(\mathbf{P}_0) = \iint_{\sigma} q(\mathbf{P}) \frac{1}{|\mathbf{P} - \mathbf{P}_0|} d\sigma - \iint_{\sigma} \varphi(\mathbf{P}) \frac{\partial}{\partial n} \frac{1}{|\mathbf{P} - \mathbf{P}_0|} d\sigma; \quad \sigma = S_1 \cup S_0, \tag{2}$$

where \mathbf{P}, \mathbf{P}_0 are points on the surface σ ; $|\mathbf{P} - \mathbf{P}_0|$ is the Cartesian distance between points \mathbf{P} and \mathbf{P}_0 ; $q(\mathbf{P})$ is the normal derivative of a function $\varphi(\mathbf{P})$ on the surface σ .

For the shell of rotation, the domain Ω , occupied by a fluid, is a body of rotation, and to solve boundary value problems (1) it is advisable to use a cylindrical coordinate system (ρ, θ, z) , in which

$$|\mathbf{P} - \mathbf{P}_0| = \sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\theta - \theta_0) + (z - z_0)^2}.$$

The following integral operator is considered

$$R(\mathbf{P}_0) = \iint_{\sigma} \gamma(P) G(|\mathbf{P} - \mathbf{P}_0|) d\sigma, \tag{3}$$

whose kernel depends only on the distance $|\mathbf{P} - \mathbf{P}_0|$. It is assumed that

$$\gamma(\mathbf{P}) = f(\rho, z) \cos(l\theta).$$

If Γ is the generator of the surface σ , then, by presenting the double integral in (3) as an iterated integral at the point $\mathbf{P}_0=(\rho_0, \theta_0, z_0)$, we will get

$$\mathbf{R}(\mathbf{P}_0) = \int_{\Gamma} f(\rho, z) \left\{ \int_0^{2\pi} \cos(l\theta) G(|\mathbf{P} - \mathbf{P}_0|) d\theta \right\} d\Gamma. \tag{4}$$

Let's make the change of variable in the inner integral in (4) $\psi=\theta-\theta_0, \theta=\psi+\theta_0$. Due to the trigonometric equality $\cos(l(\psi + \theta_0)) = \cos(l\psi)\cos(l\theta_0) - \sin(l\psi)\sin(l\theta_0)$ and 2π are periodicity of the functions in the inner integral in (4), we obtain

$$\begin{aligned} \int_0^{2\pi} \sin(l\theta) G(|\mathbf{P} - \mathbf{P}_0|) d\theta &= \int_{-\pi}^{\pi} \sin(l\theta) G(|\mathbf{P} - \mathbf{P}_0|) d\theta = 0; \\ \int_0^{2\pi} \cos(l\theta) G(|\mathbf{P} - \mathbf{P}_0|) d\theta &= \cos(l\theta_0) \int_0^{2\pi} \cos(l\psi) G(|\mathbf{P} - \mathbf{P}_0|) d\psi. \end{aligned}$$

From here

$$\mathbf{R}(\mathbf{P}_0) = f_1(\rho_0, z_0) \cos(l\theta_0),$$

where

$$f_1(\rho_0, z_0) = \int_{\Gamma} f(\rho, z) \left\{ \int_0^{2\pi} \cos(l\psi) G(|\mathbf{P} - \mathbf{P}_0|) d\psi \right\} d\Gamma.$$

Thus, the integral operators in (2) are 2π -periodic. This allows us to expand the unknown functions into Fourier series, namely, to apply the expansion

$$\Phi(\rho, \theta, z, t) = \sum_{k=1}^n \sum_{l=0}^m \dot{d}_{kl}(t) \varphi_{kl}(\rho, z) \cos(l\theta); \tag{5}$$

$$\zeta(\rho, \theta, t) = \sum_{k=1}^n \sum_{l=0}^m d_{kl}(t) \zeta_{kl}(\rho) \cos(l\theta). \tag{6}$$

Taking into account these expansions, we reduce the boundary value problem (1) to a system of one-dimensional singular integral equations in the form [10]

$$2\pi\varphi(\rho(z_0), z_0) + \int_{\Gamma} \varphi(\rho(z), z) \Theta(z, z_0) \rho(z) d\Gamma - \chi^2 \int_0^{R_0} \varphi(\rho, 0) \Phi(\mathbf{P}, \mathbf{P}) \rho d\rho = 0, \mathbf{P}_0 \in S_1; \tag{7}$$

$$2\pi\varphi(\rho, 0) + \int_{\Gamma} \varphi(\rho(z), H) \Theta(z, z_0) \rho(z) d\Gamma - \chi^2 \int_0^{R_0} \varphi(\rho, H) \Phi(\mathbf{P}, \mathbf{P}) \rho d\rho = 0, \mathbf{P}_0 \in S_0.$$

Here R_0 is the radius of free surface;

$$\Theta(z, z_0) = \frac{4}{\sqrt{a+b}} \left\{ \frac{1}{2r} \left[\frac{r^2 - r_0^2 + (z_0 - z)^2}{a-b} E_l(k) - F_l(k) \right] n_r + \frac{z_0 - z}{a-b} E_l(k) n_z \right\};$$

$$\Phi(\mathbf{P}, \mathbf{P}_0) = \frac{4}{\sqrt{a+b}} F_l(k); \quad a = \rho^2 + (z - z_0)^2; \quad b = 2\rho r_0.$$

$$E_l(k) = (-1)^l (1 - 4l^2) \int_0^{\pi/2} \cos(2l\theta) \sqrt{1 - k^2 \sin^2 \theta} d\theta; \quad F_l(k) = (-1)^l \int_0^{\pi/2} \frac{\cos(2l\theta)}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta; \quad k^2 = \frac{2b}{a+b}.$$

At values of k close to 1, we have the asymptotic expansion of the function $F_0(k)$ [23]

$$F_0(k) = \ln \frac{4}{k'} + \left(\frac{1}{2}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1.2}\right) k'^2 + \left(\frac{1.3}{2.4}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4}\right) k'^4 + \dots; \quad k'^2 = 1 - k^2,$$

from which follows the logarithmic nature of the kernel $\Phi(\mathbf{P}, \mathbf{P}_0)$ feature. Next, we consider the kernel $\Theta(z, z_0)$. Let's introduce the function

$$Q(z, z_0) = \frac{n_r r_0^2 - r^2 + (z_0 - z)^2}{2r(a-b)} + \frac{z_0 - z}{a-b} n_z$$

and obtain

$$Q(z, z_0) \approx \frac{1}{(z_0 - z)(1 + (r'_0)^2)} \left[\frac{n_r [2r_0 r'_0 + (z_0 - z)]}{2r_0} + n_z \right].$$

Since $n_r r'_0 + n_z = 0$, then at $z \rightarrow z_0$ we get

$$Q(z, z_0) \rightarrow \frac{n_r}{2r_0(1 + (r'_0)^2)} = \frac{1}{2r_0(\sqrt{1 + (r'_0)^2})^3}.$$

Thus, the kernel $\Theta(z, z_0)$ also has a logarithmic nature of the singularity.

For the numerical solution of system (7) the methods developed in [7, 21] were used. To take into account the baffles, the methods proposed in [10, 11] were applied. In [22] the boundary element superelement method was introduced. This approach is especially effective in the study of composite shells, shells with baffles, as well as shells with a branched meridian. In such cases, interface surfaces that divide the computational domain into a certain number of sub-domains are introduced. In each of these sub-domains, the velocity potential and its normal derivative are the desired functions. Compatibility conditions are imposed on the interface surfaces.

Next, the oscillations of shells with baffles, partially filled with fluid, are studied using this approach. The fluid domain is divided into two sub-domains Ω_1 and Ω_2 (Fig. 1, c). To separate these sub-domains, an artificial interface is introduced S_{int} . S_{11} and S_{12} indicate the areas of the shell surface in contact with the fluid in the domains Ω_1 and Ω_2 respectively, (Fig. 1, c). Thus, the boundaries of the sub-domains have the following form: $\sigma_1 = \partial\Omega_1 = S_{11} \cup S_{\text{int}}$ and $\sigma_2 = \partial\Omega_2 = S_{12} \cup S_{\text{int}} \cup S_0$. It should be noted that $S_{\text{baf}} \subset S_{11}$, $S_{\text{baf}} \subset S_{12}$ and $S_{\text{bot}} \subset S_{11}$.

φ_1 , φ_2 and φ_0 are potential value at nodes of the surfaces σ_1 , σ_2 and S_0 , respectively. The fluxes on the surfaces σ_1 and σ_2 are known from the impermeability condition, while on the free surface the flux is unknown and is denoted by q_0 . The values of the potential and flux on the interface S_{int} are also unknown and are denoted by φ_{ij} and q_j , $S_{\text{int}} \subset \Sigma_j, j=1, 2$ respectively. The compatibility conditions are formulated as [22]

$$\varphi_{2i} = \varphi_{1i}; \quad q_1 = -q_2.$$

We introduce integral operators of two types

$$A(S, \sigma)\psi = \iint_S \psi \frac{\partial}{\partial n} \frac{1}{|P - P_0|} dS; \quad B(S, \sigma)\psi = \iint_S \psi \frac{1}{|P - P_0|} dS, \quad P_0 \in \sigma.$$

Introducing surfaces $\tilde{S}_1 = S_{11}$, $\tilde{S}_2 = S_{\text{int}}$, $\tilde{S}_3 = S_{12}$, $\tilde{S}_4 = S_0$ allows to determine

$$A_{ij} = A(\tilde{S}_i, \tilde{S}_j); \quad B_{ij} = A(\tilde{S}_i, \tilde{S}_j), \quad i, j = \overline{1, 4}.$$

Next, to calculate the potential Φ , the boundary element superelement method is used [10], [22], which leads to a system of integral equations in operator form

$$\begin{aligned} A_{11}\varphi_1 + A_{12}\varphi_{1i} &= B_{12}q_1, \quad P_0 \in S_{11}; \\ A_{21}\varphi_1 + A_{22}\varphi_{1i} &= B_{22}q_1, \quad P_0 \in S_{\text{int}}; \\ A_{22}\varphi_{1i} + A_{23}\varphi_2 + A_{24}\varphi_0 - \chi^2 B_{24}\varphi_0 &= -B_{22}q_1, \quad P_0 \in S_{\text{int}}; \\ A_{32}\varphi_{1i} + A_{33}\varphi_2 + A_{34}\varphi_0 - \chi^2 B_{34}\varphi_0 &= -B_{32}q_1, \quad P_0 \in S_{12}; \\ A_{42}\varphi_{1i} + A_{43}\varphi_2 + A_{44}\varphi_0 - \chi^2 B_{44}\varphi_0 &= -B_{42}q_1, \quad P_0 \in S_0. \end{aligned} \tag{9}$$

We reduce system (9) to the solution of the eigenvalue problem in a similar way [10]

$$\mathbf{A}\{\varphi_0\} - \chi^2 \mathbf{B}\{\varphi_0\} = 0, \tag{10}$$

where operators \mathbf{A} and \mathbf{B} are determined by A_{ij} and B_{ij} just like in [10].

Solution system of differential equations

As a result of solving the spectral boundary value problem (1) using the integral image (2) and the boundary element superelement method, the eigenfrequencies χ_{kl} and corresponding modes φ_{kl} of fluid oscillations in a rigid tank, where $k = \overline{1, n}$, $l = \overline{0, m}$, were determined. Therefore, the expansions (5) and (6) can be used to find the unknown functions – the velocity potential Φ and the free surface elevation height ζ . To analyze the fluid motion in the shells of rotation, the pressure p is determined by the following relation [24]:

$$\frac{p}{\rho_l} = -\frac{\partial\Phi}{\partial t} - (g + a_v(t))z + a_h(t)x + \frac{p_0}{\rho_l}, \tag{11}$$

where g , $a_h(t)$, i $a_v(t)$ is the acceleration of free fall, as well as acceleration in the horizontal and vertical directions; ρ_l is the fluid density.

From equation (11) we obtain a non-homogeneous system of ordinary differential equations of the second order – the system of Mathieu differential equations [26]

$$\begin{aligned} \ddot{d}_{k0}(t) + 2\omega_{k0}\xi\dot{d}_{k0}(t) + \omega_{k0}^2\left(1 + \frac{a_v(t)}{g}\right)d_{k0}(t) &= 0; \\ \ddot{d}_{k1}(t) + 2\omega_{k1}\xi\dot{d}_{k1}(t) + \omega_{k1}^2\left(1 + \frac{a_v(t)}{g}\right)d_{k1}(t) + a_h(t)F_{k1} &= 0; \quad F_{k1} = \frac{(r, \varphi_{k1})}{(\varphi_{k1}, \varphi_{k1})}; \\ \ddot{d}_{kl}(t) + 2\omega_{kl}\xi\dot{d}_{kl}(t) + \omega_{kl}^2\left(1 + \frac{a_v(t)}{g}\right)d_{kl}(t) &= 0, \quad k = \overline{1, n}, l = \overline{2, m}, \end{aligned} \tag{12}$$

which we supplement with initial conditions in the form

$$d_{kl}(0) = d_{kl}^0; \quad \dot{d}_{kl}(0) = \dot{d}_{kl}^1, \quad k = \overline{1, n}, l = \overline{0, m}. \tag{13}$$

The system of differential equations (12) with initial conditions (13) describes the dynamic behavior of the fluid in the shell under the conditions of the joint action of vertical and horizontal excitations. In the equations of motion (12) with respect to the modal coordinates $d_{kl}(t)$, which describe the oscillations of the fluid free surface, to simulate minimal dissipative effects, artificial damping proportional to the velocity is added to the initial form without damping $2\omega_{kl}\xi\dot{d}_{kl}(t)$, where ξ is the relative damping ratio, [27]. In the following, damping in the ideal fluid model is used with the coefficient $\xi=0.005-0.01$ (0.5–1%), which meets the recommendations of ACI 350.3-06 and Eurocode 8 for fluid splash modes in tanks [27].

Numerical analysis of fluid oscillations in a rigid spherical tank

As a test example, a spherical shell without baffles of radius $R=1$ m, is partially filled with an ideal incompressible fluid to a depth of H_1 (Fig. 1, a). To solve the eigenvalue problem (10), boundary elements with a constant density approximation were used [10]. A different number of boundary elements was chosen along the generating line (N_r) and the radius of the free surface (N_0). Table 1 shows the results of determining the frequencies of axially symmetric oscillations of the fluid in the specified tank at different filling levels H_1 .

As shown in the results in Table 1, the proposed boundary element superelement method demonstrates convergence with increasing number of boundary elements. Further increase in their number does not lead to significant changes in the results. The obtained data are in good agreement with the results given in [25].

Table 1. Frequencies of axially symmetric oscillations ($l=0$) of the fluid free surface in a spherical tank

k	Method	Filling level H_1 , m					
		0.2	0.6	1.0	1.4	1.8	1.99
1	[25]	3.8261	3.6501	3.7452	4.2452	6.7645	29.0505
	BEM, $N_r=60, N_0=30$	3.8302	3.6563	3.7409	4.2409	6.7634	29.0281
	BEM, $N_r=130, N_0=65$	3.8267	3.6510	3.7449	4.2445	6.7640	29.0389
	BEM, $N_r=260, N_0=130$	3.8262	3.6502	3.7451	4.2450	6.7644	29.0495
2	[25]	9.2569	7.2652	6.9763	10.0128	12.1131	51.8123
	BEM, $N_r=60, N_0=30$	9.2672	7.2681	6.9774	10.0157	12.1237	52.2536
	BEM, $N_r=130, N_0=65$	9.2571	7.2662	6.9770	10.0144	12.1228	52.4500
	BEM, $N_r=260, N_0=130$	9.2670	7.2653	6.9766	10.0130	12.1130	51.8151

Next, we will study a spherical shell with a radius of 1 m, with a filling level of $H=1,4$ m, in which the baffle is located at the level $H_1=1,0$ m. Internal horizontal baffles are considered, which are characterized by the radius R_{int} of the interface surface; the baffle radius $R_b=R-R_{int}$.

The first three natural frequencies for mode $l=1$ are calculated for values $R_{int}=0.2$ m, $R_{int}=0.7$ m and $R_{int}=1.0$ m. It should be noted that the value $R_{int}=1.0$ m corresponds to a tank without a baffle. The obtained frequency values are given in Table 2.

Along the generator and radius of the free surface, the following values of the number of elements were chosen: $N_r=260$ and $N_\theta=130$ respectively.

The next step is to find out the required number of series terms in the expansions (5), (6) to achieve the required accuracy. It turned out that to achieve the accuracy $\varepsilon=10^{-3}$, three series terms in these expansions are enough. We will consider a spherical shell $R=1$ m without a baffle and with a baffle installed at a height $H_1=1.0$ m, with a filling level of $H=1.4$ m and $R_{int}=0.2$ m under the action of horizontal and vertical loads. We assume that

$$a_h(t) = a_0 \cos(\omega_0 t); \quad a_v(t) = a_1 \cos(\omega_1 t). \quad (14)$$

For numerical analysis, we select the values of the load parameters (14) given in Table 3.

All loads, the parameters of which are given in Table 3, lead to an unlimited increase in the amplitude of the free surface with time. This phenomenon is not physically permissible; such oscillations arise as a result of assumptions that simplify the real picture. For more substantiated conclusions, one should involve the theory of viscous fluid motion, or consider nonlinear formulations in the theory of potential motion of an ideal incompressible fluid. However, the obtained solutions, based on the assumptions of the linear theory, make it possible to provide a preliminary assessment of the stability of fluid motion in the shells, to perform detuning from undesirable resonant frequencies. Figs. 2–5 show exactly such results that correspond to the loss of stability of fluid motion in the tank. The numbers 1 denote the curves (black lines) corresponding to the fluid motion in the tank without a baffle, the numbers 2 – with a baffle installed in the spherical shell at the level $H_1=1.0$ m (blue lines), the numbers 3 correspond to the data (green lines) obtained when applying Rayleigh damping. In these calculations, $\xi=0.005$ was chosen, which in [27] is interpreted as real damping.

Fig. 2 shows the change in the level of the fluid free surface with time under resonance conditions, which occurs when the frequency of the horizontal disturbance coincides with the first natural frequency of the zeroth harmonic (case A). This mode leads to a significant increase in the amplitude of oscillations of the free surface and can significantly affect the dynamic behavior of the coupled "tank-fluid" system.

Figs. 3 and 4 show the time dependences of the fluid free surface elevation for the cases of so-called subresonances, which arise when the sum or difference of the frequencies of the vertical and horizontal perturbations coincides with the first natural frequency of the fluid oscillations.

In Fig. 3, curve 1 (black line) corresponds to the loading case when the difference between the frequencies of the vertical and horizontal disturbances is equal to the first natural frequency (case B). In contrast, in Fig. 4, curve 1 corresponds to the case when the sum of the frequencies of the vertical and horizontal disturbances is equal to the first natural frequency (case C). In both cases, a noticeable linear increase in the am-

Table 2. Frequencies of non-axisymmetric oscillations ($l=1$) of the fluid free surface in a spherical tank with various baffles

k	$R_{int}=0.2$ m	$R_{int}=0.7$ m	$R_{int}=1.0$ m
1	1.4234	2.0436	2.1234
2	5.8405	5.9723	5.9807
3	9.4567	9.4785	9.4789

Table 3. Load parameters

Load	a_h	a_v	ω_h	ω_v
A	0.6	1.0	3.7453	1.0000
B	0.6	1.0	1.0000	3.1234
C	0.6	1.0	1.0000	1.1234
D	0.0	1.0	1.0000	4.2468

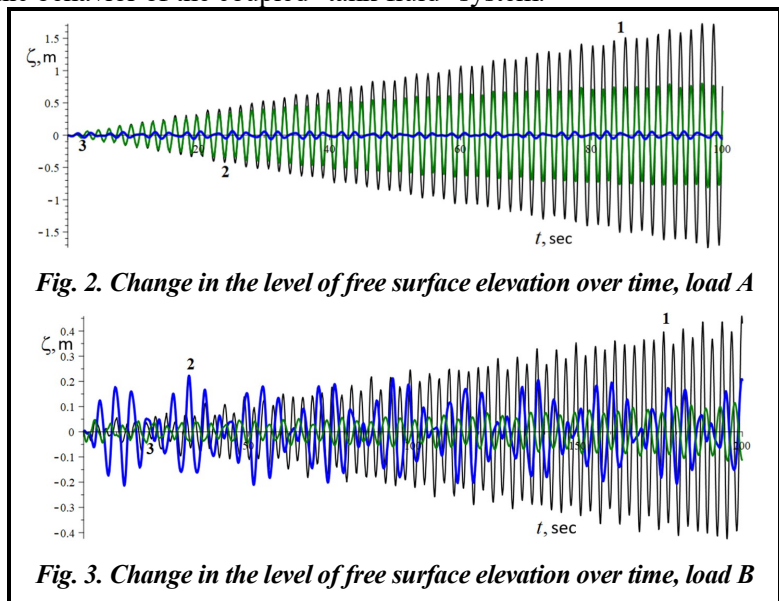


Fig. 2. Change in the level of free surface elevation over time, load A

Fig. 3. Change in the level of free surface elevation over time, load B

plitude of the free surface oscillations is observed, which indicates the possibility of significant fluid splashing under the conditions of combined disturbance.

Fig. 5 shows the dependence of the fluid free surface elevation on time under the conditions of parametric resonance. Such a regime occurs when the frequency of the vertical disturbance is equal to twice the first natural frequency of the fluid oscillations in the rigid tank, which corresponds to the first harmonic. Under these conditions, a pronounced exponential increase in the amplitude of the oscillations is observed (case D), which is a characteristic feature of parametric instability.

Thus, the fluid oscillations in spherical tanks were studied both without horizontal baffles and with their presence. It was shown that the installation of a horizontal baffle significantly reduces the level of the fluid free surface elevation. To take into account energy losses, Rayleigh damping, which is based on the use of empirical coefficients, was applied. It was established

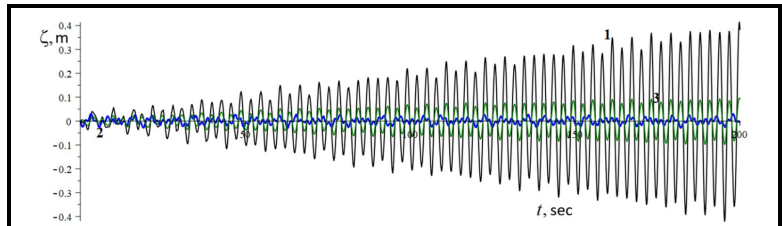


Fig. 4. Change in the level of free surface elevation over time, load C

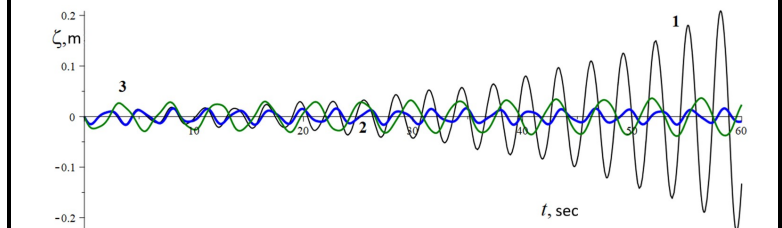


Fig. 5. Change in the level of free surface elevation over time, load D

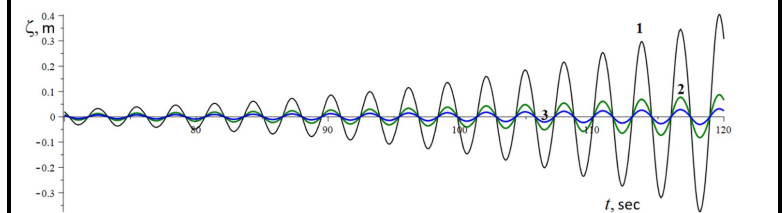


Fig. 6. Change in the level of free surface elevation over time (t from 70 to 120) sec, load D

that in tanks without baffles the amplitude of oscillations of the fluid free surface decreases, but over time, even in the presence of Rayleigh damping, its slight increase is observed. Fig. 6 shows the change in the level of the free surface in a tank without a baffle.

Fig. 6 shows the effect of artificial damping on the dynamics of the fluid free surface in a spherical tank during parametric resonance.

Curves 1, 2 and 3 correspond to the values $\xi=0$ (black), $\xi=0.005$ (green) and $\xi=0.01$ (blue). The introduction of Rayleigh damping leads to a noticeable decrease in the amplitudes of oscillations at the initial stages of the process. However, in all cases without baffles, the amplitude continues to grow with time, which indicates the dominance of the parametric mechanism of instability.

Only the installation of horizontal baffles ensures the transition to a stationary periodic regime with limited and stable amplitudes of oscillations, which confirms their effectiveness as a means of passive damping of splashes.

The obtained results reveal the mechanisms of resonant and subresonant excitation of fluid splashes. Similar effects play an important role in the dynamic behavior of partially filled tanks subjected to complex perturbations. Therefore, the presented analysis can be useful when designing and assessing the safety of fuel tanks, as well as industrial tanks intended for the storage and transportation of hazardous fluids.

Conclusions

An effective numerical approach to the study of oscillations and stability of fluid motion in rigid spherical tanks partially filled with fluid has been developed and implemented. Based on the theory of potential fluid motion and the boundary element method, spectral boundary value problems have been solved to determine the natural frequencies and forms of fluid oscillations in both spherical tanks without baffles and in tanks with horizontal baffles. The obtained forms of oscillations have been used as basis functions in the study of forced oscillations. The dynamic problem has been reduced to a system of ordinary differential equations of the Mathieu type, which made it possible to study the domains of stability of fluid motion in spherical tanks under combined loading conditions. To take into account dissipative effects in the model, Rayleigh damping based on the application of empirical coefficients has been used. Numerical results that

characterize the occurrence of parametric resonance, subresonant modes, and resonance caused by horizontal acceleration have been obtained. The features of the development of oscillations of the fluid free surface in the specified modes have been shown. The proposed approach can be used for virtual testing of spherical tanks, as well as for analyzing fluid behavior during the design and operation of tanks and fuel tanks in aerospace engineering. Further research will be aimed at studying these resonance phenomena within the framework of refined mathematical models.

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Аналіз демпфування коливань рідини в сферичних резервуарах з використанням методу граничних елементів

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Сферичні резервуари, частково заповнені рідиною, є важливими елементами сучасної техніки. Вони широко використовуються як ємності для зберігання питної води, небезпечних рідин, а також як частини паливних баків ракет-носіїв. Експериментальні випробування таких резервуарів на міцність і стійкість руху, як правило, є дороговартісними й не завжди безпечними. Це зумовлює необхідність розроблення методів віртуальних випробувань, заснованих на ефективних комп'ютерних алгоритмах. У зв'язку з цим актуальною є задача створення нових числових методів для аналізу коливань і стійкості руху рідини в резервуарах, у яких радіус вільної поверхні залежить від рівня заповнення. Виходячи з цього, мета дослідження полягає в розробленні числових методів аналізу стійкості руху у сферичних резервуарах за наявності горизонтальної перегородки. У роботі застосовано методи теорії потенціалу, граничних елементів, заданих нормальних форм, а також числові методи розв'язання систем диференціальних рівнянь. Розв'язано спектральні крайові задачі для визначення власних частот і форм коливань рідини у сферичних резервуарах без перегородок і з горизонтальними перегородками, що мають отвори різного діаметра. Ці задачі зведено до систем одновимірних сингулярних інтегральних рівнянь. Отримані власні форми коливань використано як базисні функції для розв'язання задачі про вимушені коливання рідини у сферичних резервуарах під дією одночасних вертикальних і горизонтальних збуджень. Отримано вирази для потенціалу швидкостей та функції підйому вільної поверхні у формі рядів. Проведено аналіз збіжності цих рядів. Задачу визначення динамічних характеристик рідини зведено до розв'язання системи звичайних диференціальних рівнянь типу Мат'є, що дало змогу дослідити стійкість руху рідини у сферичному резервуарі при комбінованих горизонтальних і вертикальних навантаженнях. У роботі розроблено й застосовано ефективний числовий підхід до дослідження коливань і стійкості руху рідини у сферичних резервуарах, частково заповнених рідиною. Запропонований підхід може бути використаний для віртуальних випробувань сферичних резервуарів й аналізу поведінки рідини при проектуванні й експлуатації резервуарів і паливних баків аерокосмічної техніки.

Ключові слова: коливання рідини, демпфування, сферичні резервуари, сингулярні інтегральні рівняння, горизонтальні перегородки, метод граничних елементів, стійкість руху рідини.

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