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DEVELOPMENT OF V. L. RVACHEV'S NON-ARCHIMEDEAN CALCULUS

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This paper was prepared by students of Academician V. L. Rvachev, the creator of non-Archimedean calculus. Structurally, the paper is divided into three sections. The first section examines the relativistic formula for n-ary addition. Transformations of this formula yield four results: two pairs of mutually inverse functions which, given their properties, define the construction of non-Archimedean calculus; the structure of a relativistic series; and two mappings between the sets $\Re=(-\infty, \infty)$ and $x=(-c, c)$. The second section presents relativistic trigonometry, including formulas for addition, the conversion of a sum of trigonometric functions into a product, and the conversion of a product of trigonometric functions into a sum. In the third section, relativistic hyperbolic functions are derived with the use of the relativistic exponential function, and their properties are outlined.

***Keywords**: non-Archimedean arithmetic, arithmetic with a maximal number, non-Archimedean algebra, finite infinity, relativistic sequence of natural numbers.*

Dedicated to the 100th anniversary of the birth of our Teacher – Academician V. L. Rvachev

Introduction

Nowadays, in the natural sciences – and above all in physics – problems concerning counting arise when a variable within a given set (for example, the set of natural numbers) grows beyond all bounds. Indeed, well-known problems, such as divergences in relativistic quantum mechanics, singularities in the black hole theory, and the "beginning" appearing in various cosmological models, make the extrapolation concept of counting toward quantities lying beyond the scope of empirical validation one of the most critical challenges. Clearly, significant modifications must be introduced into the logic and methodology of the post-non-classical stage of scientific development as a whole.

Within the natural sciences, a fundamental intellectual "intervention" is required. This applies not only to the search for adequate geometric models of space-time relations in the micro- and macro-worlds – an endeavor pursued for quite some time, with varying degrees of success, using non-Euclidean geometries – but also (and perhaps primarily) to the establishment of foundations for adequate arithmetic models dealing with infinitely large (or small) values. Undoubtedly, any new model claiming validity must be consistent with the established classical Archimedean model of arithmetic; it must coincide with the latter within the domain of practical counting – a requirement that reflects the well-known in the natural sciences "correspondence principle". This problem was first addressed by P. K. Rashevskij in [1].

A specific realization of the concept of such a hypothetical sequence of natural numbers, constructed as a formal mathematical apparatus for counting based on a modified Archimedean axiom, was proposed by V. L. Rvachev [2–6]. This model of non-Archimedean arithmetic arguably represents the first constructive implementation, which is relevant to the natural sciences, that moves beyond the dogmas associated with the classical sequence of natural numbers. In this model, the author utilizes the relativistic velocity-addition theorem from special relativity to construct, among other things, arithmetic operations based on the assumption that there exists a finite number c beyond which no further numbers exist. An initial analysis of the model reveals that it provides a specific mathematical representation of a numerical sequence possessing the ontological character of an actual infinity – specifically, a realization of the Hegel's "finite infinity" in the form of a limiting number. This, in turn, suggests the potential for significant progress in addressing the problems (or paradoxes) within Cantor's set theory that arise from the concept of actual infinity.

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Thus, it may be said that the methodologically justified approach initiated by P. K. Rashevskij and first implemented in the studies of V. L. Rvachev, the essence of which consists in introducing actual (finite) infinity into the axiomatic framework of arithmetic, not only provides a new impetus for the search for an adequate description of reality in the realm of the infinitely large, but also points to a concrete path toward constructing an ontological model of actual (finite) zero in the realm of infinitesimal quantities.

1. Consequences of the *n*-ary addition formula and their applications

The starting point for writing this section is relativistic *n*-ary addition [2], which is represented by the formula

$$\sum_{i=1}^{n}{}^{\circ} x_i = x_1 \overset{\circ}{+} x_2 \overset{\circ}{+} \ldots \overset{\circ}{+} x_n = \frac{(1+\alpha x_1)(1+\alpha x_2)\ldots(1+\alpha x_n)-(1-\alpha x_1)(1-\alpha x_2)\ldots(1-\alpha x_n)}{\alpha[(1+\alpha x_1)(1+\alpha x_2)\ldots(1+\alpha x_n)+(1-\alpha x_1)(1-\alpha x_2)\ldots(1-\alpha x_n)]}. \tag{1}$$

Here

$$x \overset{\circ}{+} y = \frac{x+y}{1+\alpha^2 xy};\ \alpha = c^{-1} \text{ – relativistic addition;} \tag{2}$$

c is the largest number satisfying the MAXI condition: for $|x| \le c$, $|y| \le c$ we have $x \overset{\circ}{+} y \le c$.

It should be noted that the paper retains all the notations introduced by V. L. Rvachev when presenting the theory of non-Archimedean calculus [2–4].

We assume that in (1) $x_1=x_2=\ldots=x_n=x$, then we have

$$\underbrace{x \overset{\circ}{+} x \overset{\circ}{+} \ldots \overset{\circ}{+} x}_{n \text{ times}} = M[N(x)^n]. \tag{3}$$

Corollary 1.

$$M(x) = \frac{1}{\alpha}\frac{x-1}{x+1};\ N(x) = \frac{1+\alpha x}{1-\alpha x}. \tag{4}$$

It should be noted that the functions $M(x)$ and $N(x)$ are mutually inverse, that is,

$$M(N(x)) = N(M(x)) = x. \tag{5}$$

Next, it is assumed that x=1; then we obtain

$$\overset{\circ}{n} = M[(N(1))^n], \tag{6}$$

where $\underbrace{\overset{\circ}{n} = 1 \overset{\circ}{+} 1 \overset{\circ}{+} \ldots \overset{\circ}{+} 1}_{n \text{ times}}$.

The algorithm for determining the terms of the relativistic natural sequence is considered. The first two terms, $\overset{\circ}{0}$ and $\overset{\circ}{1}$, are obtained by substituting n=0 and n=1, respectively, into formula (6). Through a simple calculation, we get

Corollary 2. $\overset{\circ}{0} = 0$, $\overset{\circ}{1} = 1$.

Subsequent terms of the sequence are obtained using the following sequence of formulas [2]

$$\overset{\circ}{2} = 1 \overset{\circ}{+} 1;\quad \overset{\circ}{3} = \overset{\circ}{2} \overset{\circ}{+} 1;\quad \overbrace{n \overset{\circ}{+} 1}^{\circ} = \overset{\circ}{n} \overset{\circ}{+} 1, \ldots$$

and, finally, we write down the relativistic natural sequence

$$\overset{\circ}{N} = \left(0, 1, \overset{\circ}{2}, \overset{\circ}{3}, \ldots, \overset{\circ}{n}, \ldots\right). \tag{8}$$

It should be noted that for α=0, series (8) coincides with the standard sequence of natural numbers

$$N = (0,1,2,3,\ldots,n,\ldots). \tag{9}$$

Formula (6) is inverted by expressing n in terms of $\overset{\circ}{n}$ using the following sequence of transformations

$$N(\overset{\circ}{n}) = [N(1)]^n;\quad \ln N(\overset{\circ}{n}) = n \ln N(1);\quad n = \gamma \ln N(\overset{\circ}{n}), \tag{10}$$

where $\gamma = \ln^{-1} N(1)$.

The left and right sides of the third formula from (10) are divided by γ and the following is obtained

$$\frac{n}{\gamma} = \ln N(\overset{o}{n}). \tag{11}$$

The result of exponentiating expression (11) is written using the formula

$$\exp\left(\frac{n}{\gamma}\right) = N(\overset{o}{n}).$$

Based on property (5), the last formula takes the form

$$\overset{o}{n} = M\left(\exp\frac{n}{\gamma}\right). \tag{12}$$

Formulas (10) and (12) are valid for any values of *n*, not just integers.

Thus, formulas (10) and (12) define ($x=n$, $\overset{o}{x} = \overset{o}{n}$)

Corollary 3.

$$\overset{o}{x} = M\left(\exp\frac{x}{\gamma}\right); \quad x = \gamma \ln N(\overset{o}{x}). \tag{13}$$

The first formula (13) is a representation of $\Re=(-\infty, \infty)$ on $x=(-c, c)$, the second one – the mapping inverse to it $x\to\Re$, where $M(x)$ and $N(x)$ are determined by formulas (4).

We denote the right-hand side of the first formula (13) as $\nu(x)$. The function $\tau(x)$, which is the inverse of $\nu(x)$, is found by carrying out the sequence

$$\nu(\tau(x)) = M\left(\exp\frac{\tau(x)}{\gamma}\right) = x; \quad \exp\left(\frac{\tau(x)}{\gamma}\right) = N(x); \quad \frac{\tau(x)}{\gamma} = \ln N(x).$$

As a result, we obtain the following

Corollary 4.

$$\nu(x) = M\left(\exp\frac{x}{\gamma}\right); \quad \tau(x) = \gamma \ln N(x), \tag{14}$$

where $\gamma = \ln^{-1} N(1)$.

It should be noted that the functions $\nu(x)$ and $\tau(x)$ are mutually inverse

$$\nu(\tau(x)) = \tau(\nu(x)) = x \ \text{(або } \nu\tau x = \tau\nu x = x)$$

and satisfy the conditions

$$\nu(0) = 0; \quad \nu(1) = 1; \quad \tau(0) = 0; \quad \tau(1) = 1. \tag{15}$$

Both functions $\nu(x)$ and $\tau(x)$ are monotonically increasing on the interval $x=(-c, c)$.

The formula

$$x \overset{o}{*} y = \nu[\tau(x) * \tau(y)] \tag{16}$$

is used to obtain relativistic operations. Here, * represents one of the four arithmetic operations, and $\overset{o}{*}$ – the corresponding relativistic operation.

1) Relativistic addition

It is showed that $\nu[\tau(x) + \tau(y)] = \dfrac{x+y}{1+\alpha^2 xy}$. According to formulas (4), (14) we obtain

$$x \overset{o}{+} y = \nu[\tau(x) + \tau(y)] = M\left[\exp\left(\frac{1}{\gamma}(\gamma \ln N(x) + \gamma \ln N(y))\right)\right] = M[\exp\ln(N(x)N(y))] = M[N(x)N(y)] =$$

$$= \frac{1}{\alpha}\left[\frac{N(x)N(y)-1}{N(x)N(y)+1}\right] = \frac{1}{\alpha}\frac{\frac{1+\alpha x}{1-\alpha x}\frac{1+\alpha y}{1-\alpha y}-1}{\frac{1+\alpha x}{1-\alpha x}\frac{1+\alpha y}{1-\alpha y}+1} = \frac{1}{\alpha}\frac{(1+\alpha x)(1+\alpha y)-(1-\alpha x)(1-\alpha y)}{(1+\alpha x)(1+\alpha y)+(1-\alpha x)(1-\alpha y)} = \frac{x+y}{1+\alpha^2 xy},$$

which corresponds to formula (2).

2) Relativistic subtraction

Using formula (16), we show that $\nu[\tau(x)-\tau(y)]=\dfrac{x-y}{1-\alpha^2 xy}$. Using the transformations of the previous paragraph, we have

$$x\overset{o}{-}y=\nu[\tau(x)-\tau(y)]=M\left[\exp\left(\frac{1}{\gamma}(\gamma\ln N(x)-\gamma\ln N(y))\right)\right]=M\left[\frac{N(x)}{N(y)}\right]=\frac{x-y}{1-\alpha^2 xy}.$$

Thus, we obtain relativistic subtraction

$$x\overset{o}{-}y=\frac{x-y}{1-\alpha^2 xy}. \tag{17}$$

Properties of operations $x\overset{o}{+}y$ and $x\overset{o}{-}y$ are given in papers [2–4], and the analysis of some of them allowed us to conclude that the operation $x\overset{o}{+}y$ defines on an interval (-*c*, *c*) an additive abelian group with a zero element 0 and an element (-*x*) opposite to *x*.

3) Relativistic multiplication

For this operation, formula (16) takes the form

$$x\circ y=\nu[\tau(x)\tau(y)]=M\left[\exp\left(\frac{1}{\gamma}((\gamma\ln N(x))(\gamma\ln N(y)))\right)\right]=M[\exp(\gamma\ln N(x)\ln N(y))]$$

or, taking into account formulas (4), we obtain

$$x\circ y=\frac{1}{\alpha}\left\{\frac{\exp\left[\gamma\ln\left|\frac{1+\alpha x}{1-\alpha x}\right|\ln\left|\frac{1+\alpha y}{1-\alpha y}\right|\right]-1}{\exp\left[\gamma\ln\left|\frac{1+\alpha x}{1-\alpha x}\right|\ln\left|\frac{1+\alpha y}{1-\alpha y}\right|\right]+1}\right\}. \tag{18}$$

4) Relativistic division

According to formula (16), we obtain

$$x\overset{o}{_o}y=\nu\left[\frac{\tau(x)}{\tau(y)}\right]=M\left[\exp\left(\frac{1}{\gamma}\frac{\gamma\ln N(x)}{\gamma\ln N(y)}\right)\right]=M\left[\exp(\frac{1}{\gamma}\ln N(x)\ln^{-1}N(y))\right]$$

or, taking into account formulas (4), we write

$$x\overset{o}{_o}y=\frac{1}{\alpha}\left\{\frac{\exp\left[\frac{1}{\gamma}\ln\left|\frac{1+\alpha x}{1-\alpha x}\right|\ln^{-1}\left|\frac{1+\alpha y}{1-\alpha y}\right|\right]-1}{\exp\left[\frac{1}{\gamma}\ln\left|\frac{1+\alpha x}{1-\alpha x}\right|\ln^{-1}\left|\frac{1+\alpha y}{1-\alpha y}\right|\right]+1}\right\}. \tag{19}$$

Properties of operations $x\circ y$ and $x\overset{o}{_o}y$ are given in [2–4].

In [4] it was shown that the operation $x\circ y$ determines a multiplicative Abelian group on $\Re_0=(-\infty,0)\cup(0,+\infty)$, $x(x\neq 0)$. The existence of the unit follows from the following: since $\tau(1)=1$, then $x\circ 1=\nu[\tau(x)\cdot 1]=x$. The element inverse to *x* will be $x^{-1}=\dfrac{1}{\tau(x)}$.

Thus, we conclude that the system of operations $\overset{o}{\Pi}=\left(x\overset{o}{+}y,x\overset{o}{-}y,x\circ y,x\overset{o}{_o}y\right)$ defines a field on the set x=(-*c*, *c*) [2]. Isomorphism of fields $\overset{o}{\Pi}$ and $\Pi=(x+y,x-y,xy,x/y)$ is implemented via mappings (13).

2. Relativistic trigonometry

In paper [4], the relativistic sine and cosine are presented as series

$$\sin_{\circ} x = x \overset{\circ}{-} (1 \overset{\circ}{\circ} 3!) \circ x^{\overset{\circ}{3}} \overset{\circ}{+} (1 \overset{\circ}{\circ} 5!) \circ x^{\overset{\circ}{5}} \overset{\circ}{-} \ldots, \tag{20}$$

$$\cos_{\circ} x = 1 \overset{\circ}{-} (1 \overset{\circ}{\circ} 2!) \circ x^{\overset{\circ}{2}} \overset{\circ}{+} (1 \overset{\circ}{\circ} 4!) \circ x^{\overset{\circ}{4}} \overset{\circ}{-} \ldots \tag{21}$$

and it is proved, using the derivative, that

$$\sin_{\circ}^{\overset{\circ}{2}} x \overset{\circ}{+} \cos_{\circ}^{\overset{\circ}{2}} x = 1,$$

and also using series (20), (21) it is shown that

$$\sin_{\circ}(\overset{\circ}{2} \circ x) = \overset{\circ}{2} \circ \sin_{\circ} x \circ \cos_{\circ} x.$$

Further, V. L. Rvachev writes: "We leave the derivation of other formulas $\sin_{\circ} x$ and $\cos_{\circ} x$ to the readers."

This is precisely what we will undertake in this section.

1) Addition and subtraction formulas:

$$\sin_{\circ}(x \overset{\circ}{+} y) = \sin_{\circ} x \circ \cos_{\circ} y \overset{\circ}{+} \cos_{\circ} x \circ \sin_{\circ} y;$$

$$\sin_{\circ}(x \overset{\circ}{-} y) = \sin_{\circ} x \circ \cos_{\circ} y \overset{\circ}{-} \cos_{\circ} x \circ \sin_{\circ} y;$$

$$\cos_{\circ}(x \overset{\circ}{+} y) = \cos_{\circ} x \circ \cos_{\circ} y \overset{\circ}{-} \sin_{\circ} x \circ \sin_{\circ} y;$$

$$\cos_{\circ}(x \overset{\circ}{-} y) = \cos_{\circ} x \circ \cos_{\circ} y \overset{\circ}{+} \sin_{\circ} x \circ \sin_{\circ} y.$$

These four formulas form the basis for deriving the following formulas.

2) Formulas for converting the sum and difference of trigonometric functions into a product:

$$\sin_{\circ} x \overset{\circ}{+} \sin_{\circ} y = \overset{\circ}{2} \circ \sin_{\circ}\left((x \overset{\circ}{+} y) \overset{\circ}{\circ} \overset{\circ}{2} \right) \circ \cos_{\circ}\left((x \overset{\circ}{-} y) \overset{\circ}{\circ} \overset{\circ}{2} \right);$$

$$\sin_{\circ} x \overset{\circ}{-} \sin_{\circ} y = \overset{\circ}{2} \circ \sin_{\circ}\left((x \overset{\circ}{-} y) \overset{\circ}{\circ} \overset{\circ}{2} \right) \circ \cos_{\circ}\left((x \overset{\circ}{+} y) \overset{\circ}{\circ} \overset{\circ}{2} \right);$$

$$\cos_{\circ} x \overset{\circ}{+} \cos_{\circ} y = \overset{\circ}{2} \circ \cos_{\circ}\left((x \overset{\circ}{+} y) \overset{\circ}{\circ} \overset{\circ}{2} \right) \circ \cos_{\circ}\left((x \overset{\circ}{-} y) \overset{\circ}{\circ} \overset{\circ}{2} \right);$$

$$\cos_{0} x \overset{\circ}{-} \cos_{0} y = \overset{\circ}{2} \circ \sin_{0}\left((x \overset{\circ}{+} y) \overset{\circ}{\circ} \overset{\circ}{2} \right) \circ \sin_{0}\left((x \overset{\circ}{-} y) \overset{\circ}{\circ} \overset{\circ}{2} \right).$$

3) Formulas for converting trigonometric functions into sums and differences:

$$\sin_{\circ} x \circ \cos_{\circ} y = \left(\sin_{\circ}(x \overset{\circ}{+} y) \overset{\circ}{+} \sin_{\circ}(x \overset{\circ}{-} y) \right) \overset{\circ}{\circ} 2;$$

$$\cos_{\circ} x \circ \cos_{\circ} y = \left(\cos_{\circ}(x \overset{\circ}{+} y) \overset{\circ}{+} \cos_{\circ}(x \overset{\circ}{-} y) \right) \overset{\circ}{\circ} 2;$$

$$\sin_{0} x \circ \sin_{0} y = \left(\cos_{0}(x \overset{\circ}{-} y) \overset{\circ}{-} \cos_{0}(x \overset{\circ}{+} y) \right) \overset{\circ}{\circ} 2.$$

Example 1. Let us prove the validity of the first formula

$$\sin_{0}(x \overset{\circ}{+} y) = \sin_{0} x \circ \cos_{0} y \overset{\circ}{+} \cos_{0} x \circ \sin_{0} y,$$

by using series (20), (21).

The series on the left-hand side of the formula has the form

$$\sin_{\circ}(x \overset{\circ}{+} y) = x \overset{\circ}{+} y \overset{\circ}{-} (1 \overset{\circ}{\circ} 3!) \circ (x \overset{\circ}{+} y)^{\overset{\circ}{3}} \overset{\circ}{+} \ldots = x \overset{\circ}{+} y \overset{\circ}{-} (1 \overset{\circ}{\circ} 3!) \circ x^{\overset{\circ}{3}} \overset{\circ}{-} (1 \overset{\circ}{\circ} 3!) \circ y^{\overset{\circ}{3}} \overset{\circ}{-} x^{\overset{\circ}{2}} \circ y \overset{\circ}{\circ} 2 \overset{\circ}{-} x \circ y^{\overset{\circ}{2}} \overset{\circ}{\circ} 2 \overset{\circ}{+} \ldots.$$

For the right-hand side, we obtain

$$\sin_{\circ} x \circ \cos_{\circ} y + \cos_{\circ} x \circ \sin_{\circ} y = \left(\left(x \overset{\circ}{-} (1^{\circ}_{\circ} 3!) \circ x^{\overset{\circ}{3}} \overset{\circ}{+} \ldots\right) \circ \left(1 \overset{\circ}{-} y^{\overset{\circ}{2}} \overset{\circ}{\circ} 2 \overset{\circ}{+} \ldots\right) \overset{\circ}{+} \left(1 \overset{\circ}{-} x^{\overset{\circ}{2}} \overset{\circ}{\circ} 2 \overset{\circ}{+} \ldots\right) \circ \left(y \overset{\circ}{-} (1^{\circ}_{\circ} 3!) \circ y^{\overset{\circ}{3}} \overset{\circ}{+} \ldots\right)\right) =$$

$$= x \overset{\circ}{+} y \overset{\circ}{-} x \circ y^{\overset{\circ}{2}} \overset{\circ}{\circ} 2 \overset{\circ}{-} (1^{\circ}_{\circ} 3!) \circ x^{\overset{\circ}{3}} \overset{\circ}{-} (1^{\circ}_{\circ} 3!) \circ y^{\overset{\circ}{3}} - x^{\overset{\circ}{2}} \circ y \overset{\circ}{\circ} 2 \overset{\circ}{+} \ldots,$$

which is what was to be proved.

Example 2. Let us prove the formula

$$\sin_{\circ} x \overset{\circ}{+} \sin_{\circ} y = 2 \circ \sin_{\circ}\left((x \overset{\circ}{+} y)^{\circ}_{\circ} 2\right) \circ \cos_{\circ}\left((x \overset{\circ}{-} y)^{\circ}_{\circ} 2\right).$$

To do this, we write down the two components of the formula

$$\sin_{\circ}(\alpha \overset{\circ}{+} \beta) = \sin_{\circ} \alpha \circ \cos_{\circ} \beta \overset{\circ}{+} \cos_{\circ} \alpha \circ \sin_{\circ} \beta;$$

$$\sin_{\circ}(\alpha \overset{\circ}{-} \beta) = \sin_{\circ} \alpha \circ \cos_{\circ} \beta \overset{\circ}{-} \cos_{\circ} \alpha \circ \sin_{\circ} \beta,$$

and add together the written-out formulas term by term

$$\sin_{\circ}(\alpha \overset{\circ}{+} \beta) + \sin_{\circ}(\alpha \overset{\circ}{-} \beta) = 2 \circ \sin_{\circ} \alpha \circ \cos_{\circ} \beta.$$

We denote $(\alpha \overset{\circ}{+} \beta) = x$, $(\alpha \overset{\circ}{-} \beta) = y$, then $\alpha = (x \overset{\circ}{+} y)^{\circ}_{\circ} 2$, $\beta = (x \overset{\circ}{-} y)^{\circ}_{\circ} 2$. By substituting in place of α and β their expressions in terms of *x* and *y*, we obtain the original formula.

Example 3. Let us also prove the formula

$$\sin_{\circ} x \circ \sin_{\circ} y = \left(\cos_{\circ}(x \overset{\circ}{-} y) \overset{\circ}{-} \cos_{\circ}(x \overset{\circ}{+} y)\right)^{\circ}_{\circ} 2.$$

As before, we write down the two components of the formula

$$\cos_{\circ}(x \overset{\circ}{-} y) = \cos_{\circ} x \circ \cos_{\circ} y \overset{\circ}{+} \sin_{\circ} x \circ \sin_{\circ} y;$$

$$\cos_{\circ}(x \overset{\circ}{+} y) = \cos_{\circ} x \circ \cos_{\circ} y \overset{\circ}{-} \sin_{\circ} x \circ \sin_{\circ} y.$$

Subtract the formulas term by term

$$\cos_{\circ}(x \overset{\circ}{-} y) \overset{\circ}{-} \cos_{\circ}(x \overset{\circ}{+} y) = \cos_{\circ} x \circ \cos_{\circ} y \overset{\circ}{+} \sin_{\circ} x \circ \sin_{\circ} y \overset{\circ}{-} \cos_{\circ} x \circ \cos_{\circ} y \overset{\circ}{+} \sin_{\circ} x \circ \sin_{\circ} y$$

and, by simplifying the right-hand side, we obtain the original formula.

3. Relativistic hyperbolic functions

Using a relativistic function $\exp_{\circ}(x)$, it is possible to write down formulas for relativistic hyperbolic functions $\mathrm{sh}_{\circ} x$, $\mathrm{ch}_{\circ} x$, $\mathrm{th}_{\circ} x$, $\mathrm{cth}_{\circ} x$:

a) hyperbolic sine

$$\mathrm{sh}_{\circ} x = \left[\exp_{\circ} x \overset{\circ}{-} \exp_{\circ}(\overset{\circ}{-} x)\right]^{\circ}_{\circ} 2; \tag{22}$$

b) hyperbolic cosine

$$\mathrm{ch}_{\circ} x = \left[\exp_{\circ} x \overset{\circ}{+} \exp_{\circ}(\overset{\circ}{-} x)\right]^{\circ}_{\circ} 2; \tag{23}$$

c) hyperbolic tangent [3]

$$\mathrm{th}_{\circ} x = \mathrm{sh}_{\circ} x \overset{\circ}{\circ} \mathrm{ch}_{\circ} x = \left(\exp_{\circ} x \overset{\circ}{-} \exp_{\circ}(\overset{\circ}{-} x)\right)^{\circ}_{\circ} \left(\exp_{\circ} x \overset{\circ}{+} \exp_{\circ}(\overset{\circ}{-} x)\right); \tag{24}$$

d) hyperbolic cotangent

$$\mathrm{cth}_{\circ} x = \mathrm{ch}_{\circ} x \overset{\circ}{\circ} \mathrm{sh}_{\circ} x = \left(\exp_{\circ} x \overset{\circ}{+} \exp_{\circ}(\overset{\circ}{-} x)\right)^{\circ}_{\circ} \left(\exp_{\circ} x \overset{\circ}{-} \exp_{\circ}(\overset{\circ}{-} x)\right). \tag{25}$$

If we represent the relativistic exponential as a series

$$\exp_{\circ} x = 1 \overset{\circ}{+} x \overset{\circ}{+} (1_{\circ}^{\circ} 2!) \circ x^{\overset{\circ}{2}} \overset{\circ}{+} \ldots \overset{\circ}{+} (1_{\circ}^{\circ} n!) \circ x^{\overset{\circ}{n}} \overset{\circ}{+} \ldots,$$

it is possible to write the series for $\mathrm{sh}_{\circ} x$, $\mathrm{ch}_{\circ} x$ as follows:

$$\mathrm{sh}_{\circ} x = x \overset{\circ}{+} (1_{\circ}^{\circ} 3!) \circ x^{\overset{\circ}{3}} \overset{\circ}{+} (1_{\circ}^{\circ} 5!) \circ x^{\overset{\circ}{5}} \overset{\circ}{+} \ldots \tag{27}$$

$$\mathrm{ch}_{\circ} x = x \overset{\circ}{+} (1_{\circ}^{\circ} 2!) \circ x^{\overset{\circ}{2}} \overset{\circ}{+} (1_{\circ}^{\circ} 4!) \circ x^{\overset{\circ}{4}} \overset{\circ}{+} \ldots \ldots \tag{28}$$

Derivatives of relativistic hyperbolic functions

$$(\mathrm{sh}_{\circ} x)^{\circ'} = \mathrm{ch}_{\circ} x; \quad (\mathrm{ch}_{\circ} x)^{\circ'} = \mathrm{sh}_{\circ} x; \quad (\mathrm{th}_{\circ} x)^{\circ'} = 1_{\circ}^{\circ} \mathrm{ch}_{\circ}^{\overset{\circ}{2}} x; \quad (\mathrm{cth}_{\circ} x)^{\circ'} = \overset{\circ}{-} 1_{\circ}^{\circ} \mathrm{sh}_{\circ}^{\overset{\circ}{2}} x. \tag{29}$$

Fundamental relations between hyperbolic functions

$$\mathrm{ch}_{\circ}^{\overset{\circ}{2}} x \overset{\circ}{-} \mathrm{sh}_{\circ}^{\overset{\circ}{2}} x = 1; \tag{30}$$

$$\mathrm{sh}_{\circ} (\overset{\circ}{2} \circ x) = \overset{\circ}{2} \circ \mathrm{sh}_{\circ} x \circ \mathrm{ch}_{\circ} x; \tag{31}$$

$$\mathrm{ch}_{\circ} (\overset{\circ}{2} \circ x) = \mathrm{sh}_{\circ}^{\overset{\circ}{2}} x \overset{\circ}{+} \mathrm{ch}_{\circ}^{\overset{\circ}{2}} x. \tag{32}$$

1) Addition and subtraction formulas:

$$\mathrm{sh}_{\circ} (x \overset{\circ}{+} y) = \mathrm{sh}_{\circ} x \circ \mathrm{ch}_{\circ} y \overset{\circ}{+} \mathrm{ch}_{\circ} x \circ \mathrm{sh}_{\circ} y;$$

$$\mathrm{sh}_{\circ} (x \overset{\circ}{-} y) = \mathrm{sh}_{\circ} x \circ \mathrm{ch}_{\circ} y \overset{\circ}{-} \mathrm{ch}_{\circ} x \circ \mathrm{sh}_{\circ} y;$$

$$\mathrm{ch}_{\circ} (x \overset{\circ}{+} y) = \mathrm{ch}_{\circ} x \circ \mathrm{ch}_{\circ} y \overset{\circ}{+} \mathrm{sh}_{\circ} x \circ \mathrm{sh}_{\circ} y;$$

$$\mathrm{ch}_{\circ} (x \overset{\circ}{-} y) = \mathrm{ch}_{\circ} x \circ \mathrm{ch}_{\circ} y \overset{\circ}{-} \mathrm{sh}_{\circ} x \circ \mathrm{sh}_{\circ} y.$$

2) Formulas for converting a sum or difference into a product:

$$\mathrm{sh}_{\circ} x \overset{\circ}{+} \mathrm{sh}_{\circ} y = \overset{\circ}{2} \circ \mathrm{sh}_{\circ}\left((x \overset{\circ}{+} y) \overset{\circ}{:}_{\circ} \overset{\circ}{2} \right) \circ \mathrm{ch}_{\circ}\left((x \overset{\circ}{-} y) \overset{\circ}{:}_{\circ} \overset{\circ}{2} \right);$$

$$\mathrm{sh}_{\circ} x \overset{\circ}{-} \mathrm{sh}_{\circ} y = \overset{\circ}{2} \circ \mathrm{sh}_{\circ}\left((x \overset{\circ}{-} y) \overset{\circ}{:}_{\circ} \overset{\circ}{2} \right) \circ \mathrm{ch}_{\circ}\left((x \overset{\circ}{+} y) \overset{\circ}{:}_{\circ} \overset{\circ}{2} \right);$$

$$\mathrm{ch}_{\circ} x \overset{\circ}{+} \mathrm{ch}_{\circ} y = \overset{\circ}{2} \circ \mathrm{ch}_{\circ}\left((x \overset{\circ}{+} y) \overset{\circ}{:}_{\circ} \overset{\circ}{2} \right) \circ \mathrm{ch}_{\circ}\left((x \overset{\circ}{-} y) \overset{\circ}{:}_{\circ} \overset{\circ}{2} \right);$$

$$\mathrm{ch}_{\circ} x \overset{\circ}{-} \mathrm{ch}_{\circ} y = \overset{\circ}{2} \circ \mathrm{sh}_{\circ}\left((x \overset{\circ}{+} y) \overset{\circ}{:}_{\circ} \overset{\circ}{2} \right) \circ \mathrm{sh}_{\circ}\left((x \overset{\circ}{-} y) \overset{\circ}{:}_{\circ} \overset{\circ}{2} \right).$$

Example 1. Let us prove that

$$\mathrm{ch}_{\circ}^{\overset{\circ}{2}} x \overset{\circ}{-} \mathrm{sh}_{\circ}^{\overset{\circ}{2}} x = 1.$$

By differentiating a function $y = \mathrm{ch}_{\circ}^{\overset{\circ}{2}} x \overset{\circ}{-} \mathrm{sh}_{\circ}^{\overset{\circ}{2}} x$, we obtain

$$y^{\circ'} = \overset{\circ}{2} \circ \mathrm{ch}_{\circ} x \circ \mathrm{sh}_{\circ} x \overset{\circ}{-} \overset{\circ}{2} \circ \mathrm{sh}_{\circ} x \circ \mathrm{ch}_{\circ} x \equiv 0$$

with the right-hand side (30) $(\mathrm{const})^{\circ'} = 0$.

Example 2. To prove relation (31), we multiply the series for $\mathrm{sh}_{\circ} x$ and $\mathrm{ch}_{\circ} x$, then we find a series for the function $(1_{\circ}^{\circ} \overset{\circ}{2}) \circ \mathrm{sh}_{\circ} (2 \circ x)$. The latter can be obtained from series (27) by substituting $\overset{\circ}{2} \circ x$ instead of x. So, we get that $\mathrm{sh}_{\circ} (\overset{\circ}{2} \circ x) = \overset{\circ}{2} \circ \mathrm{sh}_{\circ} x \circ \mathrm{ch}_{\circ} x$.

Example 3. Relation (32) is reduced to relation (31) by differentiating both the left- and right-hand sides; the subsequent steps are the same as in Example 2.

Example 4. Let us prove the validity of the relation $\mathrm{sh}_{\circ}(x \overset{\circ}{+} y) = \mathrm{sh}_{\circ}x \circ \mathrm{ch}_{\circ}y \overset{\circ}{+} \mathrm{ch}_{\circ}x \circ \mathrm{sh}_{\circ}y$, using series (27), (28).

The series for the left-hand side has the form

$$\mathrm{sh}_{\circ}(x \overset{\circ}{+} y) = x \overset{\circ}{+} y \overset{\circ}{+} (1^{\circ}_{\circ} 3!) \circ (x \overset{\circ}{+} y)^{\overset{\circ}{3}} \overset{\circ}{+} \ldots = x \overset{\circ}{+} y \overset{\circ}{+} (1^{\circ}_{\circ} 3!) \circ x^{\overset{\circ}{3}} \overset{\circ}{+} (1^{\circ}_{\circ} 3!) \circ y^{\overset{\circ}{3}} \overset{\circ}{+} x^{\overset{\circ}{2}} \circ y \overset{\circ}{\circ} 2 \overset{\circ}{+} x \circ y^{\overset{\circ}{2}} \overset{\circ}{\circ} 2 \overset{\circ}{+} \ldots .$$

For the right-hand side, we obtain

$$\mathrm{sh}_{\circ}x \circ \mathrm{ch}_{\circ}y \overset{\circ}{+} \mathrm{ch}_{\circ}x \circ \mathrm{sh}_{\circ}y = x \overset{\circ}{+} y \overset{\circ}{+} (1^{\circ}_{\circ} 3!) \circ x^{\overset{\circ}{3}} \overset{\circ}{+} (1^{\circ}_{\circ} 3!) \circ y^{\overset{\circ}{3}} \overset{\circ}{+} x^{\overset{\circ}{2}} \circ y \overset{\circ}{\circ} 2 \overset{\circ}{+} x \circ y^{\overset{\circ}{2}} \overset{\circ}{\circ} 2 \overset{\circ}{+} \ldots ,$$

which is what was to be proved.

Example 5. Let us prove the relation

$$\mathrm{sh}_{\circ}x \overset{\circ}{+} \mathrm{sh}_{\circ}y = \overset{\circ}{2} \circ \mathrm{sh}_{\circ}\left((x \overset{\circ}{+} y) \overset{\circ}{\circ} 2 \right) \circ \mathrm{ch}_{\circ}\left((x \overset{\circ}{-} y) \overset{\circ}{\circ} 2 \right).$$

We write down two proven relations:

$$\mathrm{sh}_{\circ}(\alpha \overset{\circ}{+} \beta) = \mathrm{sh}_{\circ}\alpha \circ \mathrm{ch}_{\circ}\beta \overset{\circ}{+} \mathrm{ch}_{\circ}\alpha \circ \mathrm{sh}_{\circ}\beta ;$$

$$\mathrm{sh}_{\circ}(\alpha \overset{\circ}{-} \beta) = \mathrm{sh}_{\circ}\alpha \circ \mathrm{ch}_{\circ}\beta \overset{\circ}{-} \mathrm{ch}_{\circ}\alpha \circ \mathrm{sh}_{\circ}\beta .$$

Adding the written-out relations term by term, we obtain

$$\mathrm{sh}_{\circ}(\alpha \overset{\circ}{+} \beta) \overset{\circ}{+} \mathrm{sh}_{\circ}(\alpha \overset{\circ}{-} \beta) = \overset{\circ}{2} \circ \mathrm{sh}_{\circ}\alpha \circ \mathrm{ch}_{\circ}\beta .$$

Using the notation $\alpha \overset{\circ}{+} \beta = x$, $\alpha \overset{\circ}{-} \beta = y$, $\alpha = (x \overset{\circ}{+} y) \overset{\circ}{\circ} 2$, and $\beta = (x \overset{\circ}{-} y) \overset{\circ}{\circ} 2$, applied in Example 2 of the second section, we obtain the initial relation.

Conclusions

The relativistic formula for *n*-ary addition, proposed by V. L. Rvachev, is studied. The structure of a relativistic series – that is, a series with a maximum term – is defined. A relativistic trigonometry is presented, encompassing formulas for addition, the conversion of a sum of trigonometric functions into a product, and the conversion of a product of trigonometric functions into a sum. Utilizing the relativistic exponential function, formulas for relativistic hyperbolic functions are derived, and their properties are outlined. Certain examples of applying relativistic calculus to describe physical laws within the context of a non-Archimedean Universe are presented in papers [5, 6].

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Розвиток неархимедового обчислення В. Л. Рвачова

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Статтю підготовлено учнями академіка В. Л. Рвачова – автора неархімедового обчислення. Структурно стаття поділяється на три розділи. У першому з них досліджено релятивістську формулу n-арного додавання. Перетворення згаданої формули дозволили отримати чотири наслідки, що становлять: дві пари взаємно зворотних функцій, які з урахуванням їх властивостей визначають конструкцію неархімедового обчислення; структуру релятивістського ряду; два відображення множин $\Re=(-\infty, \infty)$ и $x=(-c, c)$. У другому розділі представлено релятивістську тригонометрію, яка включає формули: додавання; перетворення суми тригонометричних функцій на добуток; перетворення добутку тригонометричних функцій на суму. У третьому розділі за допомогою релятивістської експоненти отримано формули релятивістських гіперболічних функцій та виписано їх властивості.

***Ключові слова**: неархімедове обчислення, обчислення з максимальним числом, неархімедова алгебра, кінцева нескінченність, релятивістський натуральний ряд.*

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